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**GEORGE C. MARSHALL**

**SPACE  
FLIGHT  
CENTER**

**HUNTSVILLE, ALABAMA**

PROGRESS REPORT NO. 2  
ON STUDIES IN THE FIELDS OF  
SPACE FLIGHT AND GUIDANCE THEORY

**OTS PRICE**

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**NATIONAL AERONAUTICS AND SPACE ADMINISTRATION**

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on Studies in the Fields of

SPACE FLIGHT AND GUIDANCE THEORY

Sponsored by Aeroballistics Division of

Marshall Space Flight Center

ABSTRACT

This paper contains progress reports of NASA-sponsored studies in the areas of space flight theory and guidance theory. The studies are carried on by several universities and industrial companies. This progress report covers the period from December 1, 1961 to June 15, 1962. The technical supervisor of the contracts is W.E. Miner, Deputy Chief of the Future Projects Branch of Aeroballistics Divisions, George C. Marshall Space Flight Center.

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June 26, 1962

PROGRESS REPORT NO.2  
ON STUDIES IN THE FIELDS OF  
SPACE FLIGHT AND GUIDANCE THEORY

AEROBALLISTICS DIVISION

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GEORGE C. MARSHALL SPACE FLIGHT CENTER

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Sponsored by Aeroballistics Division of  
Marshall Space Flight Center

SUMMARY

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This paper contains progress reports of NASA-sponsored studies in the areas of space flight theory and guidance theory. The studies are carried on by several universities and industrial companies. This progress report covers the period from December 1, 1961 to June 15, 1962. The technical supervisor of the contracts is W.E. Miner, Deputy Chief of the Future Projects Branch of Aeroballistics Division, George C. Marshall Space Flight Center (MSFC).

INTRODUCTION

Compiled in this paper are 11 progress papers from 7 of the agencies working under contract to MSFC in the areas of guidance theory and space flight theory.

This is the second paper of the "Progress Reports" and covers the period from December 1, 1961 to June 15, 1962. Extensive references are made to "Progress Report No. 1." This second progress report is referred to as "report" and "Progress Report No. 1" will be referred to as the "first report" in this introduction. Information given in the first report is not repeated herein.

The reports of the various contractors will be referred to by index number as papers.

There are two parallel series of publications covering the over-all activities at MSFC in the areas of guidance theory and space flight theory. One is the series of progress reports of which this paper is the second in the series. The other is the series of

"Status Reports on Theory of Space Flight and Adaptive Guidance." These series, along with a few other special reports, give a complete picture of the immediate objectives, accomplishments, and final goals of Aeroballistics Division and associated contractors in the field of space flight theory and adaptive guidance.

Five additions to the contractor support of this effort have been made after the first report was published. The University of Alabama has completed its contract and is no longer associated.

The five additions and their fields of major interest are:

Field of Study	Institution or Company
Calculus of Variation	Vanderbilt University General Electric Corporation
Celestial Mechanics	Republic Aviation Corporation General Electric Corporation
Large Computer Exploitation	Northeastern Louisiana College
Impulse Orbital Transfer	North American Aviation, Incorporated

Each of the contractor's contributions are reviewed. This review shows the particular contribution in the over-all goal of advancing space flight and space vehicle guidance.

The first three papers of this report concern work done by Grumman Aircraft Engineering Company, and are an extension of the work presented in the first report. The first paper, by Hans K. Hinz and H. Cardner Moyer, extends the gradient scheme into three dimensions where Ephemeris data are incorporated into the deck. Earth-to-Mars trajectories are used for an application. The second paper, by Gordon Pinkham, explores a method of establishing equations upon which an Encke or variation-of-parameters computation method may be based. The third paper, by Hans K. Hinz, carries out an analysis on a simplified system model in order to better understand low-thrust maneuvers in the neighborhood of a large attracting body. Two areas of study are being explored in these papers. The first is that of refining and extending the use of the gradient method. The application of the work done in the first paper is equally applicable to high-thrust trajectories. Indeed, this work may be defined as high-thrust trajectory analysis. The second two papers attack the true low-thrust problem directly. Both attack the problem under special conditions which may or may not have future utility. The present state-of-the-art for low-thrust trajectory calculations is far from being advanced, and the above work will continue.

The fourth paper in the report, by Carlos Cavoti of General Electric Corporation, was primarily developed under company sponsorship and presents a review of calculus of variations from a classical viewpoint. It brings into focus certain of the concepts of calculus of variations and may be used as a basis for establishing methods and procedures for avoiding pitfalls and for interpreting results.

These papers conclude the work presented in the calculus of variations. The effort in this field will continue during the coming period along with those of certain in-house activities in this direction.

The fifth paper of this report, by Mary Payne and Samuel Pines of Republic Aviation Corporation, is directed toward performance work of lunar or space trajectories and toward mission criteria formulation for guidance. The paper extends former work done as listed in Reference 6 of this paper. It gives a more complete formulation of the restricted three-body problem in general perturbation theory. The directions of future work and their potentials are given in the conclusions of the paper. A successful solution to a particular class of trajectories would constitute a major advance in the field. Therefore, it is expected that each of the proposed extensions will eventually be explored.

The sixth paper of this report extends the work done by the University of Kentucky team of W.S. Krogdahl, T.J. Pignani, J.B. Wells, and J.C. Eaves to special views of the restricted three-body problem. This work is aimed at establishing background for an attempted solution of the restricted problem (even in particular cases) by direct mathematical means. This approach varies from that of the former paper in that a mathematical point of view is taken, while in the former, a classical astronomy view is taken.

These two papers conclude the contractor's presentations in celestial mechanics. Work in this field is primarily directed toward eventual mission criteria formulation.

The remaining papers of the report are on large computer exploitation. The first paper of this group is by R.J. Vance of Chrysler Corporation. It describes the system of orthonormal polynomials used for approximating the steering function and time-to-cutoff function. Comments are made on the selection of data points for the best fit of a multivariant functional approximation. Appendix II of this paper gives an extended list of references on the theoretical and practical aspects of approximating functions of many variables. The work will continue at Chrysler Corporation on developing methods for multivariant functional approximations. The next paper, by Sylvia M. Hubbard and Shigemichi Seyuki of the University of North Carolina, extends the work reported in the first report in application of linear programming to the approximation of the steering equation as used in the adaptive guidance mode. Practical steering equations are being generated at the present time by least-squares methods as discussed

by Vance. This method gives results which compare favorably with instrument and thrust decay errors. However, future advances in instrumentation and propulsion systems dictate that further control of the mathematical errors will be required. Continued work needs to be done in linear programming for this reason.

The tenth paper proposes a study of the numerical properties of functions of more than one independent variable. This proposal is made by James W. Hanson and Richard J. Painter of the University of North Carolina. The quotation with which this paper is introduced gives an expert's estimate of the state-of-the-art in multivariant functional approximations. Among the points proposed in this paper for further study, some have been followed through MSFC in-house activities. Aside from these, the points proposed should be considered.

The eleventh paper is by Daniel E. Dupree of Northeastern Louisiana State College. This paper presents sufficiency conditions for the existence of multivariable least-squares approximating functions. The theorem developed shows that there exists a least-squares solution except in very special cases. It is felt that no numerical procedure need be developed at present to check results for the satisfaction of the sufficiency conditions.

The last paper of this report is by James W. Hanson of the University of North Carolina. This paper extends work reported in the first report on "Analytical Differentiation by Computer." The extensions made to the original differentiation program are given along with proposed improvements and final objectives.

These papers conclude the work done in the field of large computer exploitation.

The work done by the contractors during the last 6 months has proceeded in an orderly manner toward the final objectives.

In conclusion, it may be noted that in spite of the formidable difficulties encountered in most of the problems attacked, the progress made within the limited time of the life of the contracts, together with the results brought about by in-house activities, laid the foundation for the application of the adaptive guidance made in the Saturn vehicle flights.

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**RESEARCH DEPARTMENT  
GRUMMAN AIRCRAFT ENGINEERING CORPORATION**

**THREE-DIMENSIONAL LOW-THRUST  
INTERPLANETARY TRAJECTORY OPTIMIZATION**

**By**

**Hans K. Hinz  
H. Gardner Moyer**

**BETHPAGE, NEW YORK**

SYMBOLS

$a, b, \dots, \ell$	elements of the transformation matrices
$A, B, C$	transformation matrices
$a_E, a_T$	semimajor axes of the earth's and target planet's orbit
$b_E, b_T$	semiminor axes of the earth's and target planet's orbit
$c$	rocket exhaust velocity
$D$	$d/10,000$
$e_E, e_T$	eccentricities of the earth's and target planet's orbit
$E_E, E_T$	eccentric anomalies of the earth and target planet
$\Delta E_E$	differential correction of the earth's eccentric anomaly
$i_T^\circ$	inclination of the target planet's orbital plane with respect to the ecliptic
$k_1 \dots k_{33}$	constants for computation of ephemeris data such that angular measures are expressed in degrees
$K_1 \dots K_7$	positive penalty constants
$m$	mass of the vehicle
$M_E^\circ, M_T^\circ$	mean anomalies of the earth and target planet, in degrees

$x_s, y_s, z_s$	vehicle's position components with respect to ecliptic coordinate system (Fig. 2)
$x_E, y_E$	earth's position components with respect to a coordinate system for which the $x_E - y_E$ axes are in the ecliptic plane and the positive $x_E$ -axis passes through the perihelion of the earth's orbit
$x_T, y_T$	target planet's position components with respect to the target planet's coordinate system
$\alpha$	angle between the thrust vector and the orbital plane of the target planet (Fig. 1)
$\beta$	angle between the vehicle's velocity vector and the ecliptic plane (Fig. 2)
$\gamma$	angle between the projection of the vehicle's velocity vector on the ecliptic plane and the normal to the radius vector (Fig. 2)
$\delta$	angle between the thrust vector and the ecliptic plane
$\zeta$	angle between the projection of the vehicle's position vector on the ecliptic plane and the vernal equinox (Fig. 2)
$\eta$	throttle control variable
$\theta$	angle between the projection of the thrust vector on the ecliptic plane and the $x$ -axis of target planet's coordinate system (Fig. 1)

$M_E$	mean anomaly of the earth, in radians
$n$	an integer used in the computation of $M_E^\circ$ and $M_T^\circ$ such that these angles are limited to values between 0 and 360 degrees
$n_E$	mean orbital frequency of the earth, in radians per unit time
$n_T^\circ$	mean orbital frequency of the target planet, in degrees per unit time
$P'$	modified function composed of terminal values to be minimized and penalty terms to be made zero
$R$	distance of the vehicle from the Sun
$t$	time
$T$	thrust of vehicle in the equations of motion and transformation equations
$T$	d/36,525 in the equations used to compute the planetary orbital elements
$T_T$	time of perihelion passage of the target planet
$u, v, w$	vehicle's velocity components with respect to the target planet's coordinate system
$V$	vehicle's velocity with respect to any heliocentric-inertial coordinate system
$x, y, z$	vehicle's position components with respect to the target planet's coordinate system (Fig. 1)

$\mu$	gravitational parameter of the Sun
$\xi$	angle between vehicle's position vector and the target planet's orbital plane (Fig. 1)
$\phi$	angle between the projection of the thrust vector on the ecliptic plane and the normal to the radius vector (Fig. 2)
$\psi$	angle between the projection of the vehicle's position vector on the target planet's orbital plane and the positive s-axis of the target planet's coordinate system
$\omega_E^\circ$	argument of perihelion of the earth's orbit, measured from the vernal equinox, in degrees
$\omega_T^\circ$	argument of perihelion of the target planet's orbit, measured from its ascending node, in degrees
$\tilde{\omega}_T^\circ$	longitude of perihelion of the target planet's orbit, measured from the vernal equinox along the ecliptic to the ascending node, and then along the orbit from the node to the perihelion, in degrees
$\Omega_T^\circ$	longitude of the ascending node of the target planet, measured from the vernal equinox, in degrees

### Superscripts

(-)	prescribed values of the constraints on the terminal conditions of the state variables
-----	--

$( )^\circ$	angle measured in degrees
$(\dot{\phantom{x}})$	derivative with respect to time
$(\ddot{\phantom{x}})$	second derivative with respect to time

### Subscripts

E	earth
$E_1, E_2$	first and second approximations of the mean or eccentric anomaly of the earth
f	terminal values
n	$n^{\text{th}}$ interval of time
o	initial value
s	ecliptic system
T	target planet
x, y, z	components of a vector in the directions of the target planet's coordinate axes (Fig. 1)
$x_s, y_s, z_s$	components of a vector in the directions of the ecliptic coordinate axes (Fig. 2)

### Other

(o)	initial condition
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RESEARCH DEPARTMENT  
GRUMMAN AIRCRAFT ENGINEERING CORPORATION  
BETHPAGE, NEW YORK

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THREE-DIMENSIONAL LOW-THRUST  
INTERPLANETARY TRAJECTORY OPTIMIZATION

by

Hans K. Hinz  
H. Gardner Moyer

Summary

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A successive approximation technique has been used to compute three-dimensional optimal low-thrust interplanetary trajectories. For this purpose two IBM 7090 computer programs have been developed — one for constant thrust applications, and the other for variable thrust problems. The computer programs generate and store the planetary ephemeris, and automatically optimize the trajectories for missions such as rendezvous and intercept. The 'penalty function' technique is used to handle constraints on the position and velocity components and the fuel weight at the terminal point.

Optimum low-thrust Earth-to-Mars rendezvous and intercept trajectories have been numerically computed for two synodic periods from January 1965 to July 1969. Initial results, when compared with previous coplanar studies, indicate that there are no highly significant differences between two- and three-dimensional optimum low-thrust

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interplanetary trajectories. This tentative conclusion, based upon limited results, should not be generally accepted until additional numerical studies are carried out. The result appears to be associated with the small (1.85 degree) inclination of Mars' orbit to the ecliptic.

### INTRODUCTION

The problem of optimizing the trajectory of a low-thrust space vehicle has been attacked by the classical variational method with only limited success. The main difficulties are associated with the two-point boundary value problem arising in numerical solution of the Euler-Lagrange equations of optimal flight, and with the inequality constraints imposed on control variables. It is believed that these difficulties present major obstacles to solution of the more realistic three-dimensional problems by the classical technique.

The use of successive approximation methods, classed among the direct methods of the calculus of variations, has been explored at Grumman in an attempt to circumvent the two-point boundary value difficulty. Indications from earlier work with this technique were sufficiently encouraging to exploit the gradient scheme further for more complex problems. More recent efforts utilizing this approach have provided a clear indication of the practical usefulness of this class of methods when employed in conjunction with high-speed digital computation.

The goal of the Low-Thrust Trajectory Optimization Project is the development of techniques and computer programs for determination of minimum time/minimum fuel trajectories for both geocentric orbital transfers and interplanetary rendezvous and fly-by operations. This report describes only the interplanetary phase of the project. We focus attention upon nominal trajectory determination and, for the present, place no emphasis upon guidance considerations, perturbation effects, or error analysis.

The first year of effort has been devoted to: application of the existing method to optimum coplanar rendezvous with Mars, further development of optimization techniques, and development of an IBM 7090 program for generating realistic three-dimensional optimum low-thrust trajectories.

In order to gain experience with the newly developed 'penalty function' technique for handling equality constraints on terminal conditions, an experimental computer program was prepared. The target planet's orbit for this program is idealized as circular and coplanar with that of the vehicle. Optimization is based upon determination of a thrust steering schedule for which the time of flight from Earth to rendezvous with Mars is minimized. Since rocket thrust for this problem is taken to be constant and continuous, minimizing time is equivalent to minimizing fuel expenditure. For a thrust/weight ratio of  $8.47 \times 10^{-5}$  the minimum transfer time varies from 197 to about 385 days, the latter corresponding to launch conditions for which the planets are in the most unfavorable relative positions.

Further development efforts have resulted in three related successive approximation schemes for treating variational problems that have inequality constraints on control variables. The three methods have been compared using as an illustration a planar Earth-Mars transfer with rocket thrust variable between zero and some specified maximum value.

In contrast to the two-dimensional investigation aimed primarily at technique development, a subsequent three-dimensional effort has as its goal realism sufficient to be useful in studies of actual missions. Thus the orbits of the planets of departure and destination are taken as elliptic and noncoplanar. It is this program, together with some preliminary numerical results for Mars rendezvous and intercept missions, which is described in this report.

## COMPUTER PROGRAMS

Two computer programs have been developed and are now operational for determining optimum three-dimensional low-thrust interplanetary trajectories. One of these programs is specialized to constant thrust, while the other deals with variable, but limited, thrust engines. With this exception the two computer programs are identical.

### System Model

Although the computer program is capable of dealing with missions initiated from any position in the solar system it will ordinarily be assumed that the vehicle has been launched from some planet of departure and boosted to a velocity sufficient for escape from that planet. Thus the initial components of position and velocity of the vehicle with respect to the heliocentric-inertial system are taken to be identical with those of the planet of departure.

It is assumed that the entire flight is under the effects of solar gravity only. It is felt that because the vehicle has been boosted to escape conditions the gravitational attraction of any near planet is a local effect, insignificant as far as performance is concerned. If it becomes significant, as in a planetary fly-by and return to earth, the gravity effects could be accounted for by an instantaneous rotation of the velocity vector with respect to the near planet, at the point of nearest approach. The deflection angle, easily derivable from a two-body analysis, would be a function of both the approach velocity, and the approach distance.

The orbits of the planets of departure and destination are taken as elliptic and noncoplanar. For each trajectory to be optimized the planetary orbital elements are computed for the date of departure, using ephemeris information (Ref. 1), and taken as constant throughout the flight. Thus sufficient realism is obtained without resorting to

tape-stored data or other lengthy computational procedures.

The low-thrust rocket engine employed for this investigation is assumed to have a constant exhaust velocity (constant specific impulse) and a thrust magnitude which is constant in the first case or variable between fixed limits in the second. Consequently, the mass of the vehicle decreases at a rate proportional to the thrust. The optimal thrust program within limits  $0 \leq T \leq T_m$  is sought via successive approximation simultaneously with the two optimal steering angle programs.

Optimization for all trajectories is based upon minimizing the time to complete a particular mission, such as rendezvous or intercept. For the constant thrust application, the fuel expended is not limited, and minimizing time is equivalent to minimizing fuel. For the variable thrust case the total propellant allocated is less than that required for the corresponding constant thrust example. Therefore one may expect to obtain coasting arcs or  $T < T_m$  for the variable thrust cases.

### System Equations

For computational efficiency we orient our heliocentric-inertial coordinate system such that the x-y plane contains the target planet's orbit, and the positive x-axis includes the perihelion of the planet. This simplifies description of the planetary motion by reducing the computations associated with the orbital elements and related transformations. Although this choice requires separate computer routines for calculating initial conditions from planetary ephemerides, and for transforming the position and velocity components of the final optimized trajectory to an ecliptic reference system, the total number of computations are reduced, and in addition the coordinates which describe the vehicle's motion are available in both the ecliptic and the target planet's coordinates.

The equations of motion in Cartesian coordinates (Fig. 1) and the equation governing change in mass are as follows:

$$\dot{u} = (T/m) \cos \alpha \cos \theta - \mu x/R^3$$

$$\dot{v} = (T/m) \cos \alpha \sin \theta - \mu y/R^3$$

$$\dot{w} = (T/m) \sin \alpha - \mu z/R^3$$

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{z} = w$$

$$\dot{m} = -T/c$$

$$T = \eta T_m$$

$$R^2 = x^2 + y^2 + z^2$$

where  $\mu$  is the gravitational parameter of the sun,  $T_m$  is the maximum value of thrust,  $c$  is the constant exhaust velocity, and  $\eta$  is the throttle control variable limited to values between zero and one. The three control variables are:  $\theta$  and  $\alpha$  (the thrust steering angles measured with respect to the target planet's coordinate system - Fig. 1), and  $\eta$ .

The successive approximation technique, including the equations and associated logic has been described in considerable detail in Refs. 2 and 3, and need not be reported in this report. One of these equations which defines  $P'$ , the function to be minimized, is worthy of further discussion:

$$P' = t_f + \frac{1}{2}K_1(x_f - \bar{x}_f)^2 + \frac{1}{2}K_2(y_f - \bar{y}_f)^2 + \frac{1}{2}K_3(z_f)^2$$

$$+ \frac{1}{2}K_4(u_f - \bar{u}_f)^2 + \frac{1}{2}K_5(v_f - \bar{v}_f)^2 + \frac{1}{2}K_6(w_f)^2 + \frac{1}{2}K_7(m_f - \bar{m}_f)^2$$

where  $t_f$  is the final time;  $K_1$  through  $K_7$  are positive penalty constants;  $\bar{m}_f$  is the prescribed limit of propellant to be consumed;  $\bar{x}_f \dots \bar{v}_f$  are the target planet's position and velocity components at final time, and  $x_f \dots m_f$  are the final values of the state variables.

For the general variable thrust rendezvous mission all penalty constants are nonzero. Although the program could be utilized to optimize a constant thrust trajectory by setting  $K_7$  equal to zero, it is more efficient to employ the specialized program in which all variable thrust features are eliminated and the order of integration reduced by two.

When  $K_4$ ,  $K_5$ , and  $K_6$  are set equal to zero, the computer program is converted to solution of the problem of intercept trajectory optimization.

### Initial Condition Program

Since the target planet's coordinate system is used to compute the trajectories, it is necessary to determine the initial conditions of the vehicle (which are assumed the same as those of the earth) in terms of these coordinates. A short separate program not only computes these initial conditions but also determines the current orbital elements of the terminal planet.

Defining  $d$  as the number of Julian days since January 0.5, 1900, the conventional orbital elements of the target planet (Ref. 1) are:

$$i_T^\circ = k_1 + k_2 T + k_3 T^2$$

$$\Omega_T^\circ = k_4 + k_5 T + k_6 T^2 + k_7 T^3$$

$$\tilde{\omega}_T^\circ = k_8 + k_9 T + k_{10} T^2 + k_{11} T^3$$

$$M_T^{\circ} = k_{12} + k_{13}d + k_{14}D^2 + k_{15}D^3 - 360 n$$

$$e_T = k_{16} + k_{17}T + k_{18}T^2$$

$$\omega_T^{\circ} = \tilde{\omega}_T^{\circ} - \Omega_T^{\circ}$$

$$T_T = k_{21}(360 - M_T^{\circ})/n_T^{\circ}$$

where  $D = d/10,000$ ;  $T = d/36,525$ ;  $n$  is an integer which limits  $M_T$  to an angle between 0 and 360 degrees, and  $k_{21}$  is a conversion factor for computing the time of perihelion passage,  $T_T$ , in the desired units of time. Superscript zero indicates that the angle is numerically expressed in degrees. The values of  $k_1$  through  $k_{18}$  and  $n_T$  are given in Ref. 1 for the inner planets (Mercury to Mars). It should be mentioned that if the target is one of the outer planets (Jupiter to Pluto) a slightly different procedure would be preferred.

Similar equations apply to the earth's orbital elements:

$$e_E = k_{23} + k_{24} D + k_{25} D^2$$

$$M_E^{\circ} = k_{26} + k_{27} d + k_{28} D^2 + k_{29} D^3 - 360 n$$

$$\omega_E^{\circ} = k_{30} + k_{31} d + k_{32} D^2 + k_{33} D^3$$

After conversion from degrees to radians the eccentric anomaly,  $E_E$ , is determined by a differential corrections method (Ref. 4):

$$E_{E_1} = M_E$$

$$M_{E_1} = E_{E_1} - e_E \sin E_{E_1}$$

$$\Delta E_{E_1} = (M_E - M_{E_1}) / (1 - e_E \cos E_{E_1})$$

$$E_{E_2} = E_{E_1} + \Delta E_{E_1}$$

The last three equations are recomputed iteratively once or twice in order to achieve an accuracy of eight significant figures.

The earth's position and velocity are then determined in a coordinate system where the  $x_E$ - $y_E$  plane is the ecliptic and the  $x_E$ -axis passes through the perihelion of the earth's orbit:

$$x_E = a_E (\cos E_E - e_E)$$

$$y_E = b_E \sin E_E$$

$$\dot{x}_E = -a_E \dot{E}_E \sin E_E$$

$$\dot{y}_E = b_E \dot{E}_E \cos E_E$$

where  $a_E$  is the semimajor axis of the earth's orbit and:

$$b_E = a_E \sqrt{1 - e_E^2}$$

$$\dot{E}_E = n_E / (1 - e_E \cos E_E)$$

The constant  $n_E$  is the mean orbital frequency of the earth.

The vehicle's position and velocity, assumed equal to the earth's, is then computed with respect to the target

planet's coordinate system. In matrix notation:

$$[x(0) \ y(0) \ z(0)] = [x_E \ y_E][B][A]$$

$$[\dot{x}(0) \ \dot{y}(0) \ \dot{z}(0)] = [\dot{x}_E \ \dot{y}_E][B][A]$$

where

$$[B] = \begin{bmatrix} k & \ell \\ -\ell & k \end{bmatrix}$$

$$[A] = \begin{bmatrix} a & d & g \\ b & e & h \end{bmatrix}$$

$$a = \cos \Omega_T \cos \omega_T - \sin \Omega_T \sin \omega_T \cos i_T$$

$$b = \sin \Omega_T \cos \omega_T + \cos \Omega_T \sin \omega_T \cos i_T$$

$$d = -\sin \Omega_T \cos \omega_T \cos i_T - \cos \Omega_T \sin \omega_T$$

$$e = \cos \Omega_T \cos \omega_T \cos i_T - \sin \Omega_T \sin \omega_T$$

$$g = \sin \Omega_T \sin i_T$$

$$h = -\cos \Omega_T \sin i_T$$

$$k = \cos \omega_E$$

$$\ell = \sin \omega_E$$

### Target Planet Motion

The target planet is assumed to move in a two-body orbit with orbital elements determined by the date of departure. Prior to trajectory optimization an ephemeris is computed, stored, and used repeatedly for each trajectory solution.

The eccentric anomaly,  $E_T$ , is computed by the same differential correction method described for the initial condition program, with the exception that instead of setting the initial guess for  $E_{T_1}$  equal to  $M_T$  advantage is taken of the results of the previous iteration cycle by using a Taylor series extrapolation:

$$E_{T_1}(t_{n+1}) = E_T(t_n) + \dot{E}(t_n)\Delta t + \frac{1}{2}\ddot{E}(t_n)\Delta t^2$$

where

$$\dot{E}_T(t_n) = n_T/[1 - e_T \cos E_T(t_n)]$$

$$\ddot{E}_T(t_n) = \dot{E}_T(t_n)^2 [M_T(t_n) - E_T(t_n)]/[1 - e_T \cos E_T(t_n)]$$

With the iterated values of  $E_T$ ,  $\dot{E}_T$ , and  $\ddot{E}_T$ , the position, velocity, and acceleration of the planet are determined from

$$x_T = a_T(\cos E_T - e_T)$$

$$y_T = b_T \sin E_T$$

$$\dot{x}_T = -a_T \dot{E}_T \sin E_T$$

$$\dot{y}_T = b_T \dot{E}_T \cos E_T$$

$$\ddot{x}_T = -a_T(\ddot{E}_T \sin E_T + \dot{E}_T^2 \cos E_T)$$

$$\ddot{y}_T = b_T(\ddot{E}_T \cos E_T - \dot{E}_T^2 \sin E_T)$$

where

$$b_T = a_T \sqrt{1 - e_T^2}.$$

The two components of acceleration are utilized to determine the second derivative of the function  $P'$ , which in turn determines the terminal time criterion.

### Transformation to Ecliptic Coordinates

A short subroutine has been programmed for transforming the vector components of position, velocity, and thrust from the target planet reference system to more conventional ecliptic coordinates, and for computing, in addition, the more descriptive angles that orient the above three vectors.

The vector transformations in matrix notation are, of course, all the same:

$$\begin{bmatrix} x_s & y_s & z_s \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix} [C]$$

$$\begin{bmatrix} \dot{x}_s & \dot{y}_s & \dot{z}_s \end{bmatrix} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix} [C]$$

$$\begin{bmatrix} T_{x_s} & T_{y_s} & T_{z_s} \end{bmatrix} = \begin{bmatrix} T_x & T_y & T_z \end{bmatrix} [C]$$

where

$$[C] = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

$$c = \sin \omega_T \sin i_T$$

$$f = \cos \omega_T \sin i_T$$

$$j = \cos i_T$$

$$T_x = \cos \alpha \cos \theta$$

$$T_y = \cos \alpha \sin \theta$$

$$T_z = \sin \alpha$$

and  $a, b, d, e, g,$  and  $h$  have been previously defined for the initial condition program. The vector components on the right side of the above equations are with respect to the target planet's coordinate system, and the transformed components on the left are with respect to the ecliptic system.

In order to describe the vehicle's motion and optimum thrust program more effectively some auxiliary angles (Figs. 1 and 2) are computed by the subroutine:

$$\phi = \arctan \left[ (x_s T_{x_s} + y_s T_{y_s}) / (x_s T_{y_s} - y_s T_{x_s}) \right]$$

$$\delta = \arcsin [T_{z_s}]$$

$$\gamma = \arctan \left[ (x_s \dot{x}_s + y_s \dot{y}_s) / (x_s \dot{y}_s - y_s \dot{x}_s) \right]$$

$$\beta = \arcsin [\dot{z}_s / V]$$

$$\psi = \arctan[y/x]$$

$$\xi = \arcsin[z/R]$$

$$\zeta = \arctan[y_s/x_s]$$

where

$$V = \sqrt{u^2 + v^2 + w^2}$$

### NUMERICAL DATA

#### Ephemeris Data

All numerical data pertaining to the solar system are taken from Ref. 1. The constants given below are for Mars, the target planet of the present investigations.

$k_1 = 1.8503333$	$k_{16} = 9.33129 \times 10^{-2}$
$k_2 = -6.75 \times 10^{-4}$	$k_{17} = 9.2064 \times 10^{-5}$
$k_3 = 1.2611111 \times 10^{-5}$	$k_{18} = -7.7 \times 10^{-8}$
$k_4 = 4.8786442 \times 10^1$	$k_{21} = 8.64 \times 10^4$
$k_5 = 7.7099167 \times 10^{-1}$	$k_{23} = 1.675104 \times 10^{-2}$
$k_6 = -1.3888889 \times 10^{-6}$	$k_{24} = -1.1444 \times 10^{-5}$
$k_7 = -5.3333333 \times 10^{-6}$	$k_{25} = -9.4 \times 10^{-9}$

$$\begin{aligned}
k_8 &= 3.342182 \times 10^2 & k_{26} &= 3.5847584 \times 10^2 \\
k_9 &= 1.8407583 & k_{27} &= 9.8560027 \times 10^{-1} \\
k_{10} &= 1.2986111 \times 10^{-4} & k_{28} &= -1.12 \times 10^{-5} \\
k_{11} &= 1.1944444 \times 10^{-6} & k_{29} &= -7 \times 10^{-8} \\
k_{12} &= 3.1952942 \times 10^2 & k_{30} &= 1.0122083 \times 10^2 \\
k_{13} &= 5.2402077 \times 10^{-1} & k_{31} &= 4.70684 \times 10^{-5} \\
k_{14} &= 1.3553 \times 10^{-5} & k_{32} &= 3.39 \times 10^{-5} \\
k_{15} &= 2.5 \times 10^{-8} & k_{33} &= 7 \times 10^{-8}
\end{aligned}$$

$$a_T = 7.4738897 \times 10^{11} \text{ ft}$$

$$n_T^\circ = 5.2403295 \times 10^{-1} \text{ deg/J.D.}$$

$$a_E = 4.9051201 \times 10^{11} \text{ ft}$$

$$n_E = 1.9909866 \times 10^{-7} \text{ rad/sec}$$

$$\mu = 4.6782736 \times 10^{21} \text{ ft}^3/\text{sec}^2$$

where  $k_{21}$  is a conversion factor (seconds per Julian day), and all other  $k$ 's have units such that the quantities  $i_T^\circ$ ,  $\Omega_T^\circ$ ,  $\tilde{\omega}_T^\circ$ ,  $M_T^\circ$ ,  $M_E^\circ$ , and  $\omega_E^\circ$  are expressed in degrees.

### Vehicle Parameters

In order to compare the significant differences between two- and three-dimensional optimum trajectories, the same vehicle parameters used for the previous coplanar studies were used in the present investigation.

Maximum Thrust, $T_m$	0.127 lb
Exhaust Velocity, $c$	$1.8306952 \times 10^5$ ft/sec
Initial Mass, $m(0)$	46.583851 slug
Initial Weight	1500 lb
Thrust/Weight	$8.467 \times 10^{-5}$
Propellant Consumption Rate	1.93 lb/day
Specific Impulse	5685 sec

### SOME COMPUTATIONAL RESULTS

#### Mars Rendezvous

Optimum Earth-to-Mars rendezvous trajectories have been computed for a series of departure dates from January 1, 1965 to July 1, 1969 for intervals of every three months. These 19 launch dates cover a range of two synodic periods (the synodic period of Mars is about 780 days, or 2.135 years). For this initial phase of the investigation, the thrust magnitude is considered constant and continuous.

Eight of these optimized trajectories, for succeeding launch dates, are presented in Figs. 3 to 10. Because of the low inclination of Mars' orbit (1.85 deg.) the  $z_s$ -component of distance is usually quite small and is therefore not shown. Instead, dashed lines are used to indicate that the vehicle's path is below the ecliptic.

When comparing Fig. 3 with Figs. 4 through 10, an abrupt change in the characteristics of the vehicle's flight path is noted. The vehicle, for Fig. 3, flies out past the Martian orbit and "waits" for the planet to overtake it. In Fig. 4 the vehicle "decides" that rather than wait for Mars it is more "profitable" in terms of time and fuel to increase its angular velocity, by passing close to the Sun, and eventually catching up with the planet. Although the transfer angle is about 360 degrees larger, the transfer time is less. The following sequence of figures (5 through 10) shows that the same scheme is employed, and that for each succeeding launch date the vehicle's closest approach to the Sun at first decreases and then increases. Eventually, in Fig. 10, the entire trajectory lies within the orbits of Earth and Mars. It is in this latter class of solutions that the minimum minimum of the minimum time rendezvous trajectories can be found. For the following synodic period the same cycle repeats itself.

It is interesting to note these characteristics are identical to those observed for the coplanar solutions. In fact, all of the results so far indicate that there are no distinct or highly significant differences between two- and three-dimensional optimum low-thrust trajectories. The perceptible differences that do exist, appear to be associated more with the eccentricity of the planets and show up as a random-type phase shift rather than a significant change in performance. The minimum and maximum transfer times are 202 and 382 days, and differ only slightly from the extremes of 197 and 385 days obtained from the coplanar studies (see Fig. 12). These conclusions are preliminary and should not be generally accepted until additional solutions are carried out, particularly near the two extremes of the performance curves.

Time histories for the two thrust steering angles are shown in Fig. 11. The angle  $\phi$ , which is the angle between the projection of the thrust vector on the ecliptic and the local horizontal of the Sun (Fig. 2), is shown to vary in a manner that is typical of other steering programs, and in particular is quite similar to the optimum solutions obtained from our linearized near-circular orbit

analysis. Usually about half-way in transit the thrust direction will reverse itself in a relatively short period of time.

The angle  $\alpha$ , which is the angle between the thrust vector and the orbital plane of Mars, is also shown in Fig. 11 and is generally quite small. This may be expected since it is this control component which must change the small inclination of the vehicle's orbital plane by 1.85 degrees without unduly penalizing the in-plane energy-producing component of the thrust vector.

Why  $\alpha$  drops to -14 degrees is presently not known. It is probably due to incomplete convergence since it occurs in the same region where  $\phi$  is changing abruptly from about 70 to 250 degrees.

### Mars Intercept

Also plotted in Fig. 12 are the performance results of some typical optimum intercept trajectories. For these examples the thrust magnitude is assumed constant. Because of the removal of the constraints upon the terminal values of the velocity components, there is an appreciable savings in transfer time — as much as 86 days. Whether or not there is a proportional savings in fuel consumed for the corresponding fly-by-and-return mission could be determined only by detailed analysis of the round-trip case.

With the exception of the significant reduction in transfer time, the intercept trajectories are characteristically quite similar to those for rendezvous. Fig. 13 is a comparison of the two types of trajectories, each launched on the same date. For most of the flight the two paths almost coincide, and only near the end is there a very noticeable divergence. This is typical of missions launched at other times as well.

### Variable Thrust Rendezvous

At present optimum rendezvous missions with variable thrust engines have been examined for only one launch date. The date selected, January 1, 1967, is close to that for which the optimum rendezvous missions with constant thrust have a performance minimum with respect to launch date.

As in the previous constant thrust case, the results indicate a similarity between two- and three-dimensional trajectory solutions. The following table briefly summarizes the significant reduction in fuel requirements that may be achieved for a 1500 pound vehicle if the transfer time is permitted to be slightly longer:

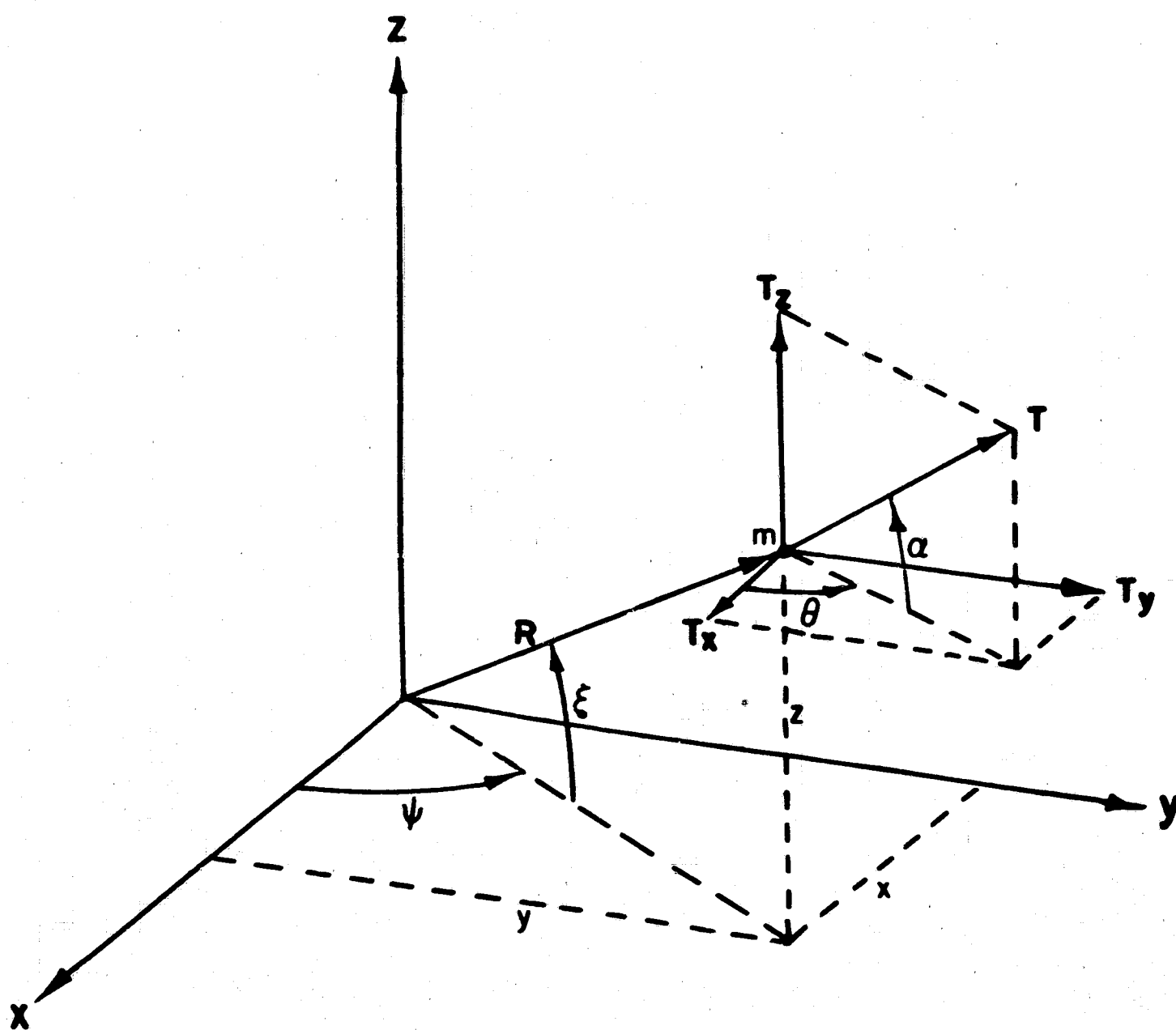
Final Weight (lb)	1095	1248	1298
Transfer Time (days)	210	226	242
First Thrust Period (days)	-	52	48
Coast Period (days)	-	44	136
Final Thrust Period (days)	-	80	58
Final Weight/Initial Weight	.730	.832	.865
% Decreased in Consumed Fuel	-	37.8	50.1

As in the coplanar studies, the thrust magnitude program converged to clearly defined regions of full thrust and coast.

### REFERENCES

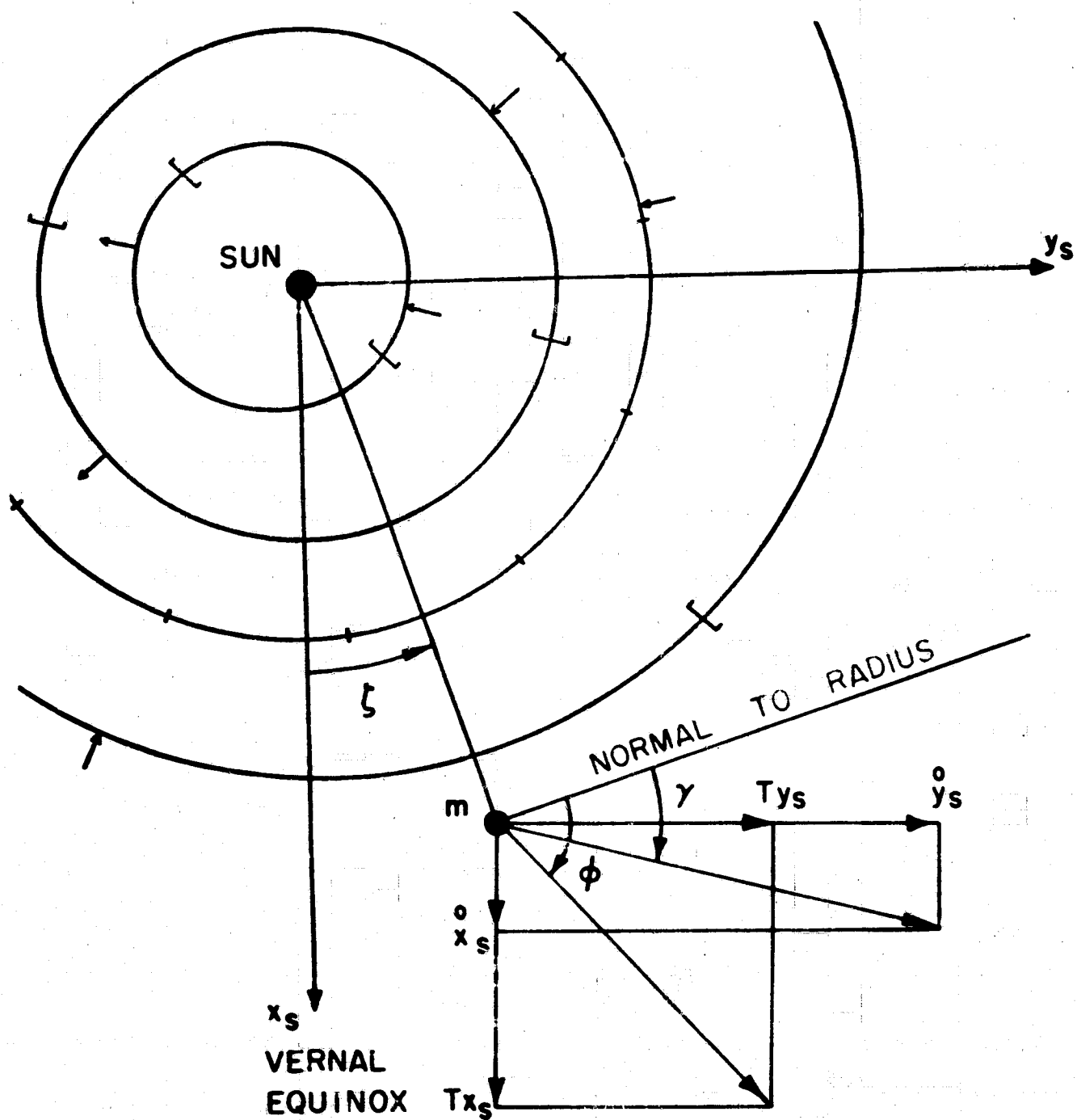
1. Explanatory Supplement to the Ephemeris, Her Majesty's Stationery Office, London, 1961.
2. Low-Thrust Trajectory Optimization, Grumman contribution to "Progress Report No. 1 On Studies in the Fields of Space Flight and Guidance Theory," NASA-Marshall Rept. MTP-Aero-61-91, December 18, 1961.
3. Kelley, H.J., Method of Gradients, Chapter 6 of "Optimization Techniques," G. Leitmann, Ed., Academic Press, to appear.
4. Moulton, F.R., Celestial Mechanics, Sect. 96, Macmillan Co., New York, 1914.

FIG. 1 TARGET PLANET COORDINATE SYSTEM



NOTES :  $X$  AND  $Y$  AXES ARE IN THE TARGET  
 PLANET'S ORBITAL PLANE — PERIHELION OF  
 PLANET'S ORBIT IS ON POSITIVE  $X$  AXIS

FIG. 2 ECLIPTIC COORDINATE SYSTEM

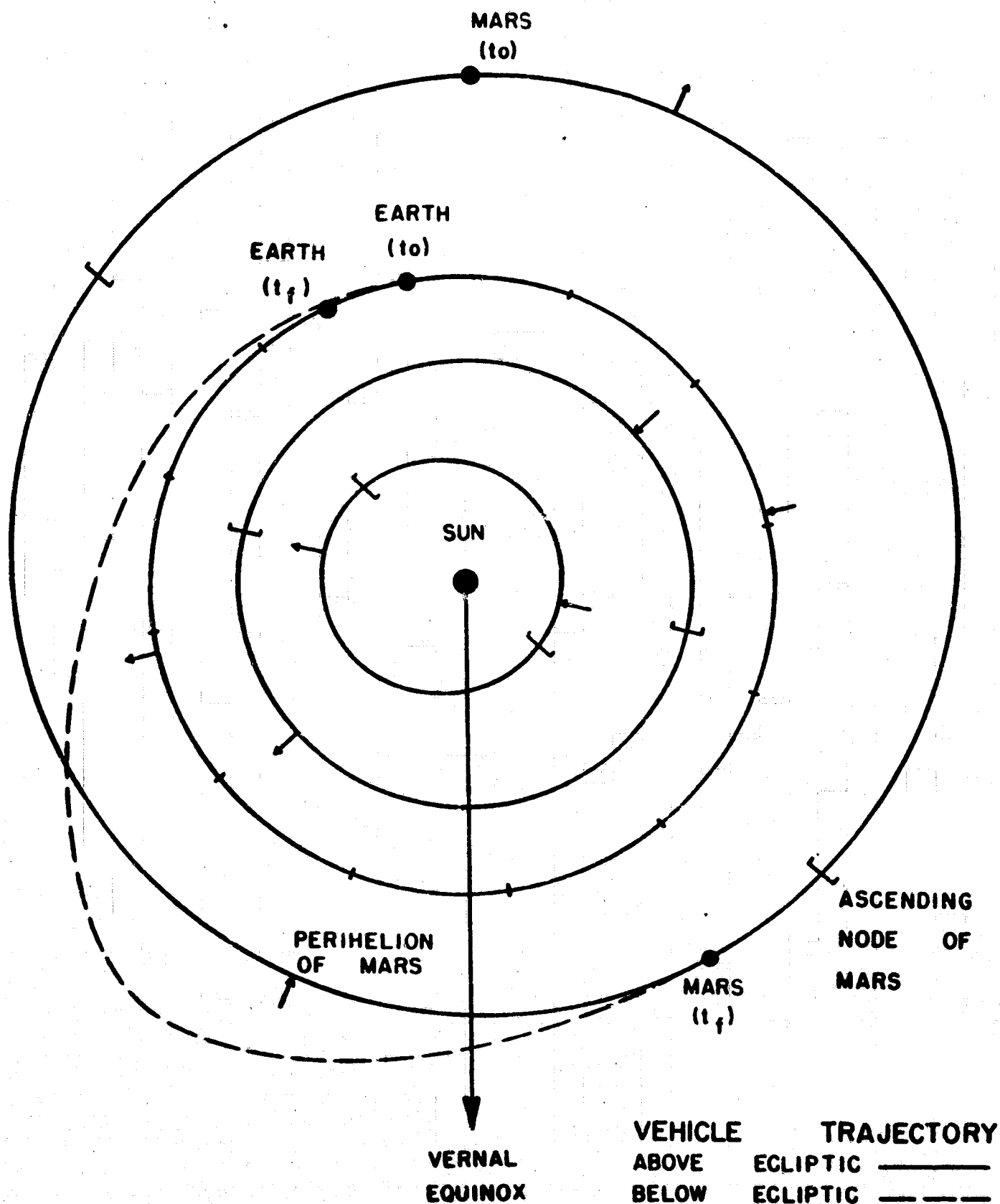


NOTES :  $x_s$  AND  $y_s$  AXES ARE IN THE  
ECLIPTIC PLANE — VERNAL EQUINOX DEFINES  
POSITIVE  $x_s$  AXIS.

# FIG. 3 OPTIMUM RENDEZVOUS TRAJECTORY

DEPARTURE DATE : APR 1 1965

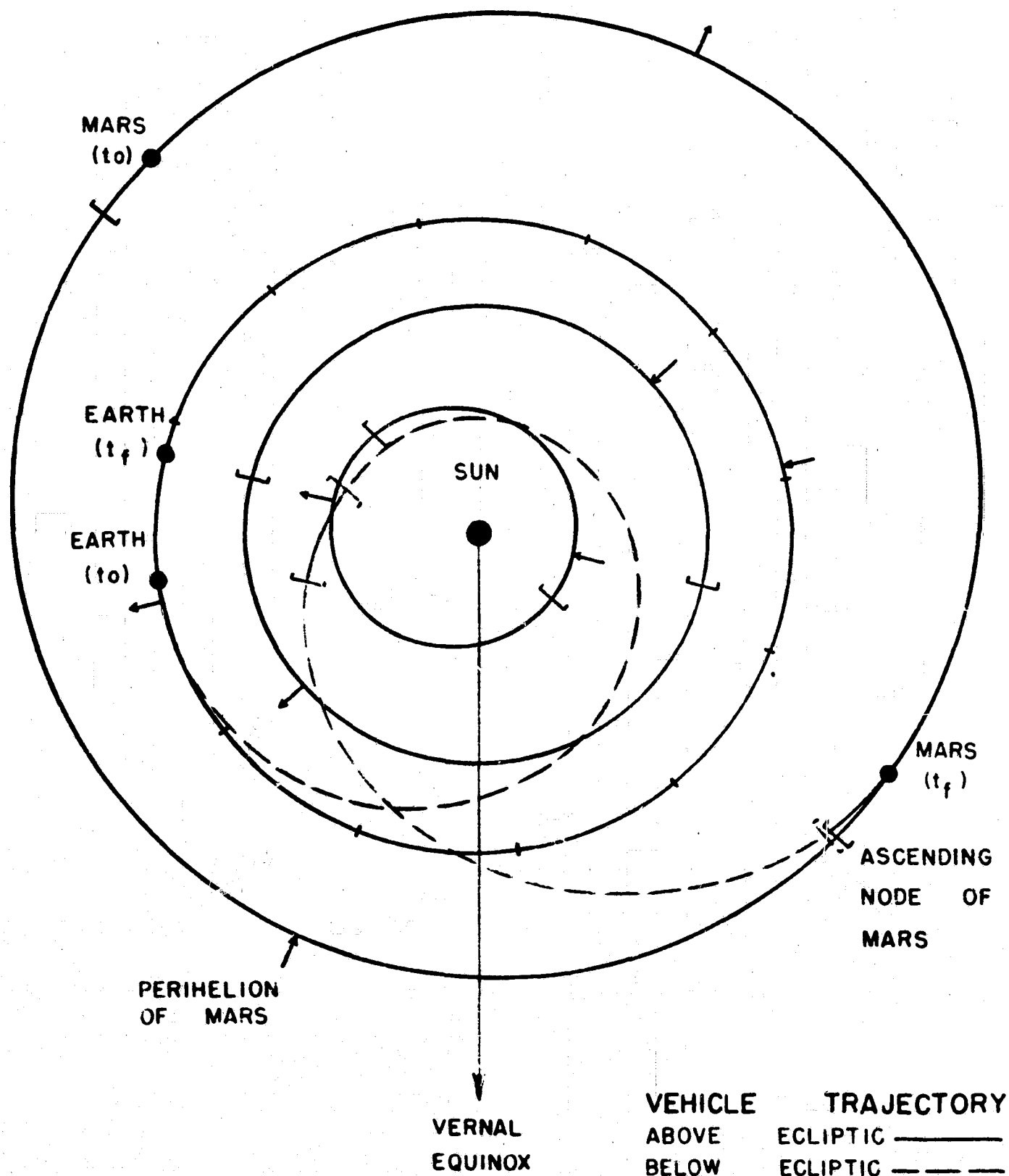
TRANSFER TIME : 382 DAYS



# FIG. 4 OPTIMUM RENDEZVOUS TRAJECTORY

DEPARTURE DATE : JUL 1 1965

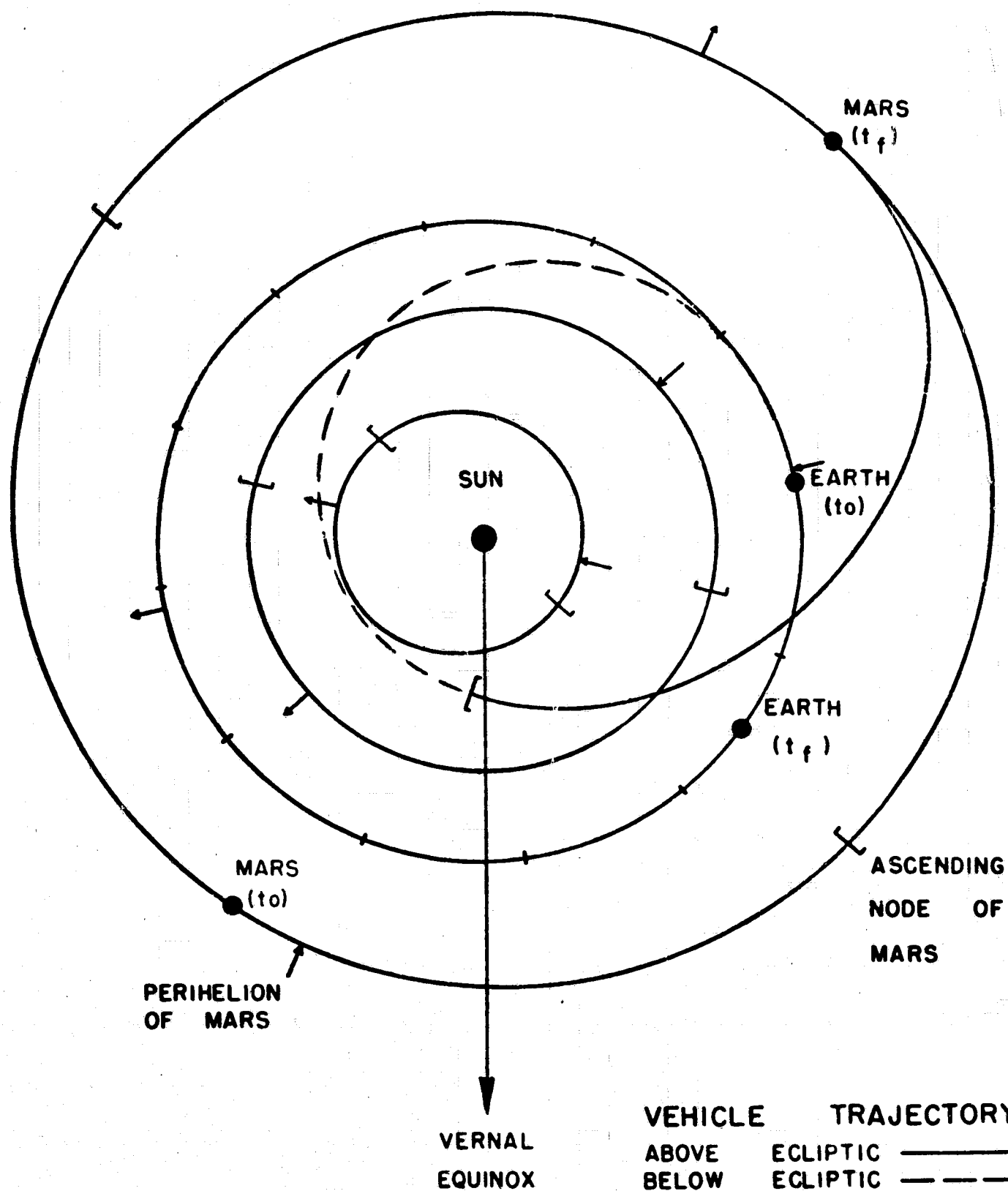
TRANSFER TIME : 340 DAYS



# FIG. 6 OPTIMUM RENDEZVOUS TRAJECTORY

DEPARTURE DATE : JAN 1 1966

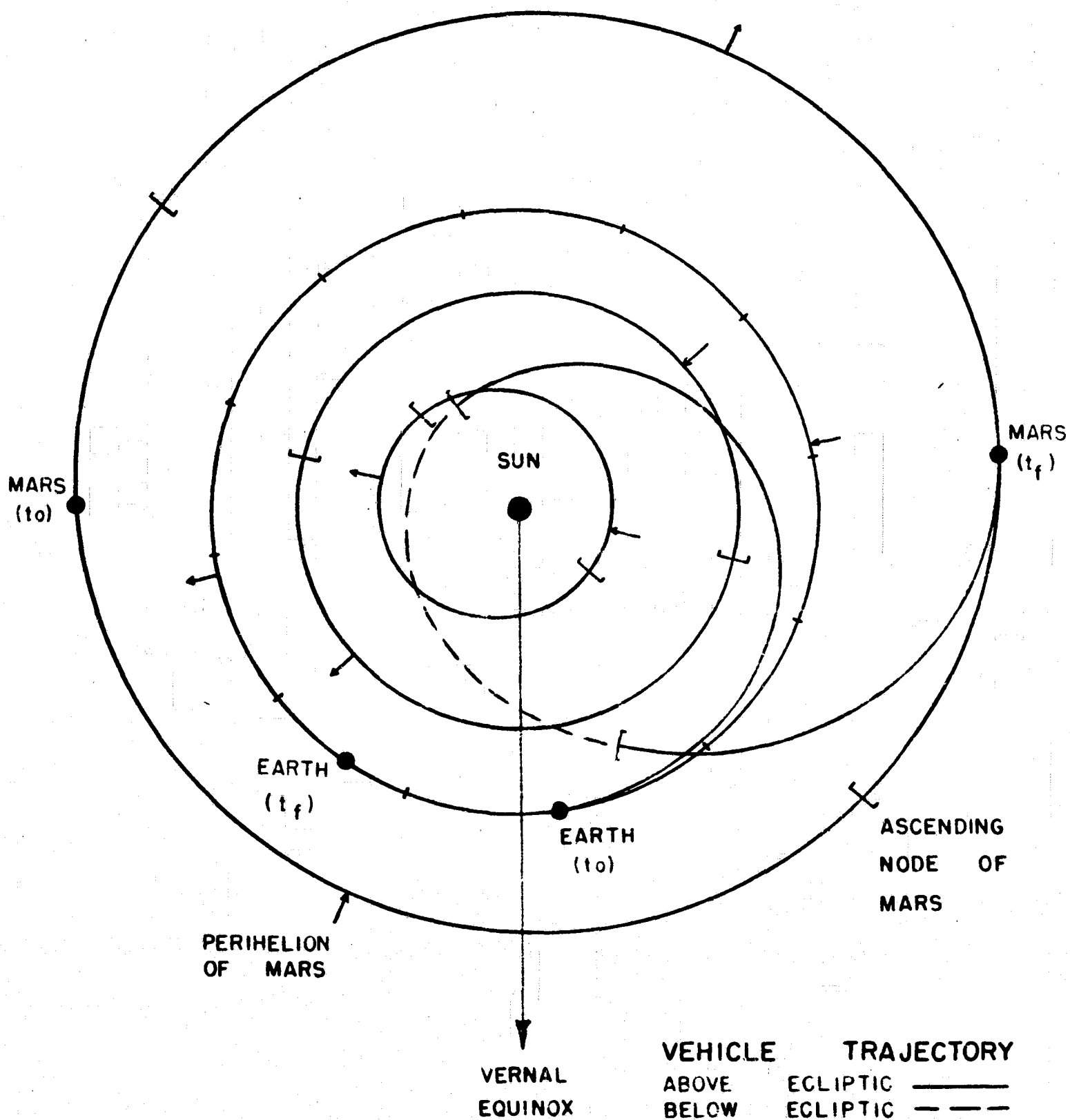
TRANSFER TIME : 318 DAYS



# FIG. 5 OPTIMUM RENDEZVOUS TRAJECTORY

DEPARTURE DATE : OCT 1 1965

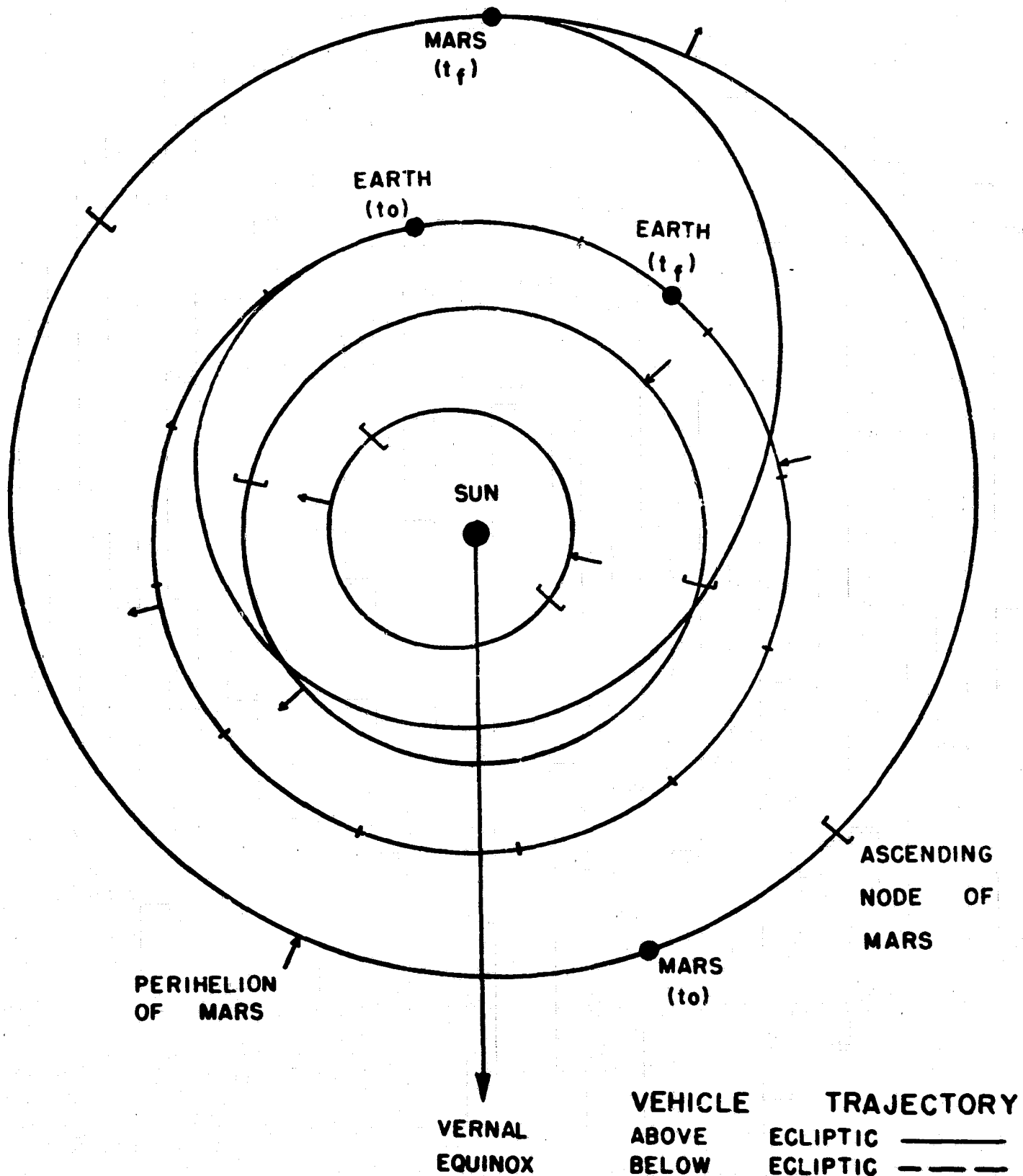
TRANSFER TIME : 320 DAYS



# FIG. 7 OPTIMUM RENDEZVOUS TRAJECTORY

DEPARTURE DATE : APR 1 1966

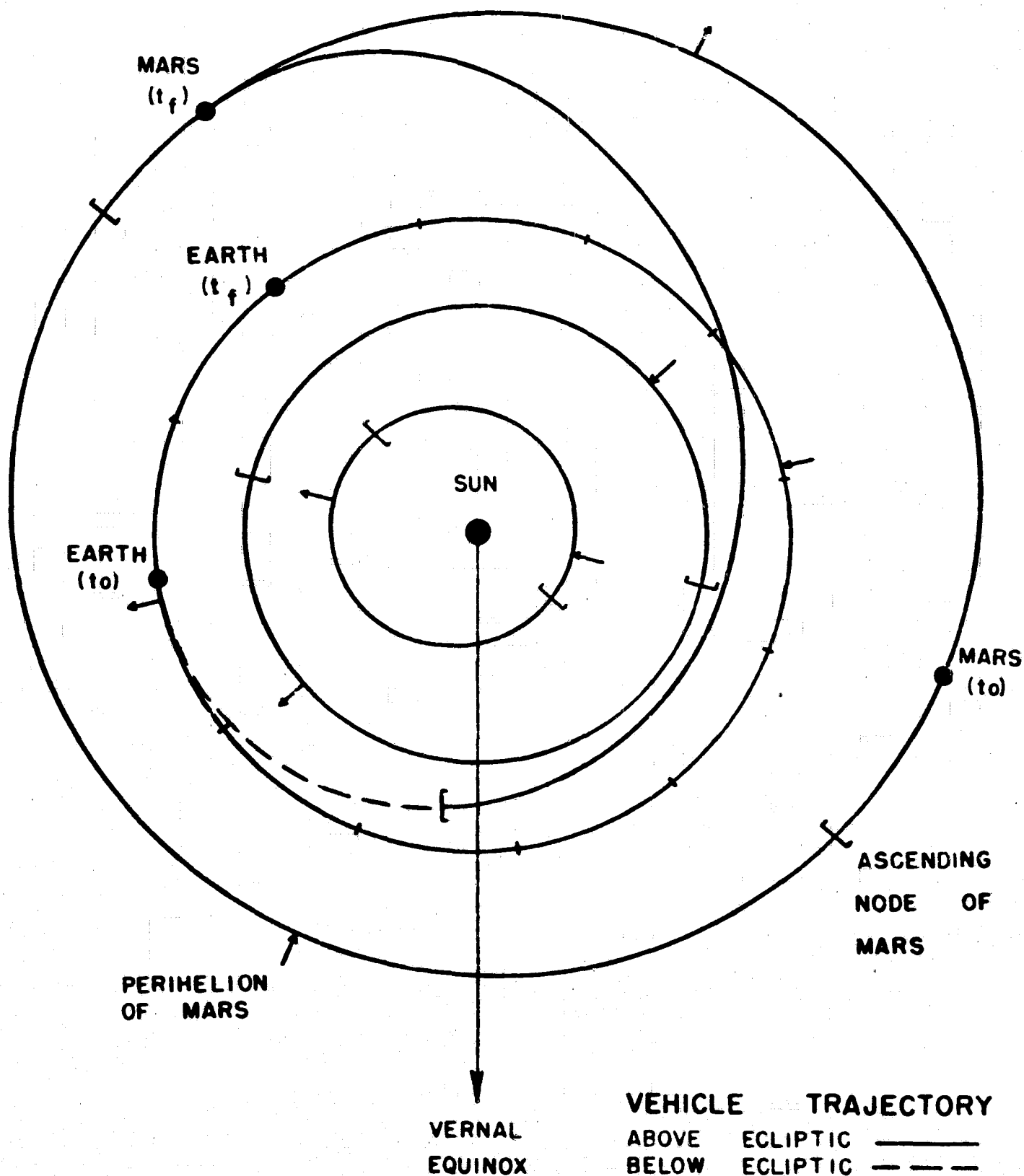
TRANSFER TIME : 314 DAYS



# FIG. 8 OPTIMUM RENDEZVOUS TRAJECTORY

DEPARTURE DATE : JUL 1 1966

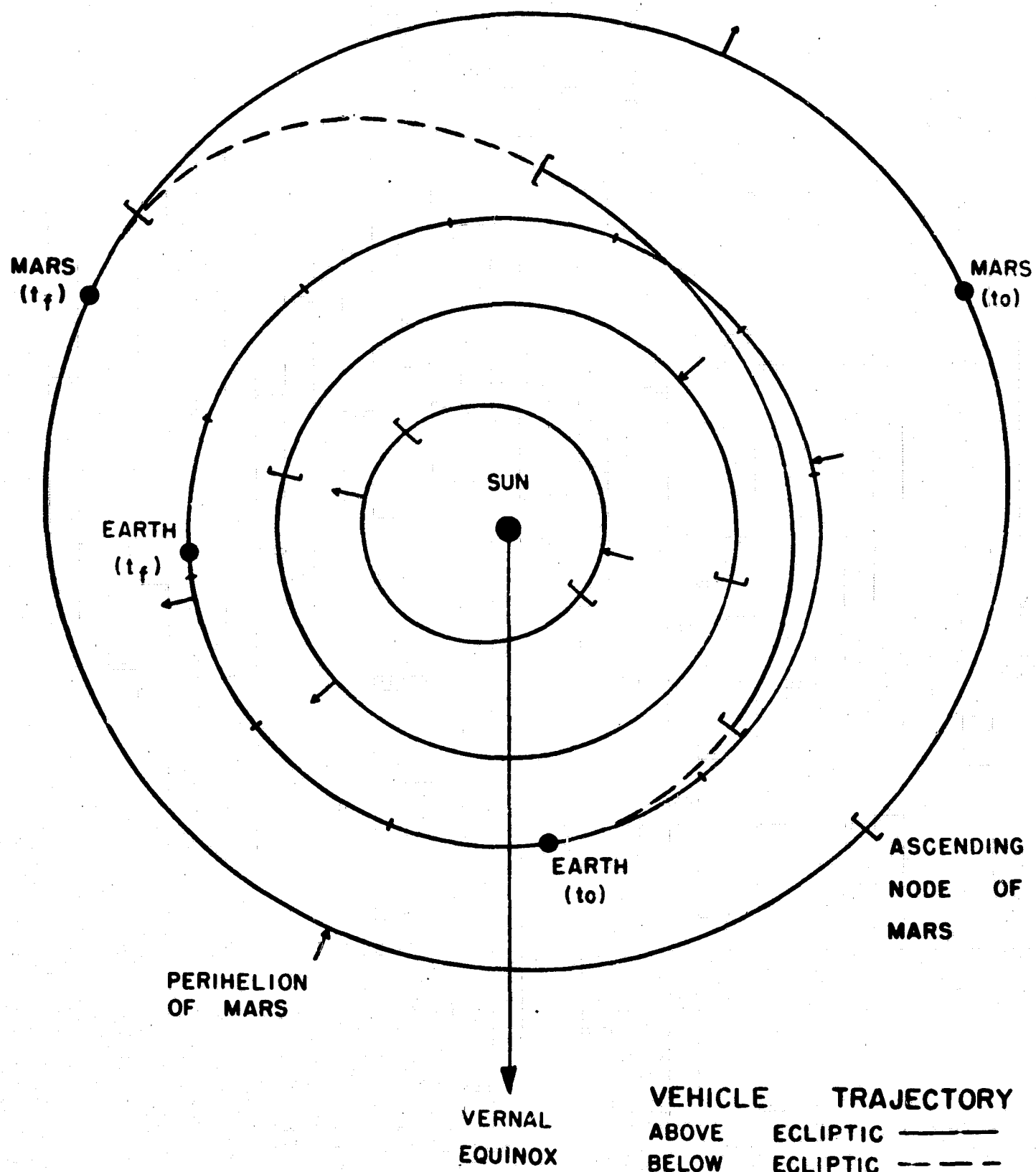
TRANSFER TIME : 302 DAYS



# FIG. 9 OPTIMUM RENDEZVOUS TRAJECTORY

DEPARTURE DATE : OCT 1 1966

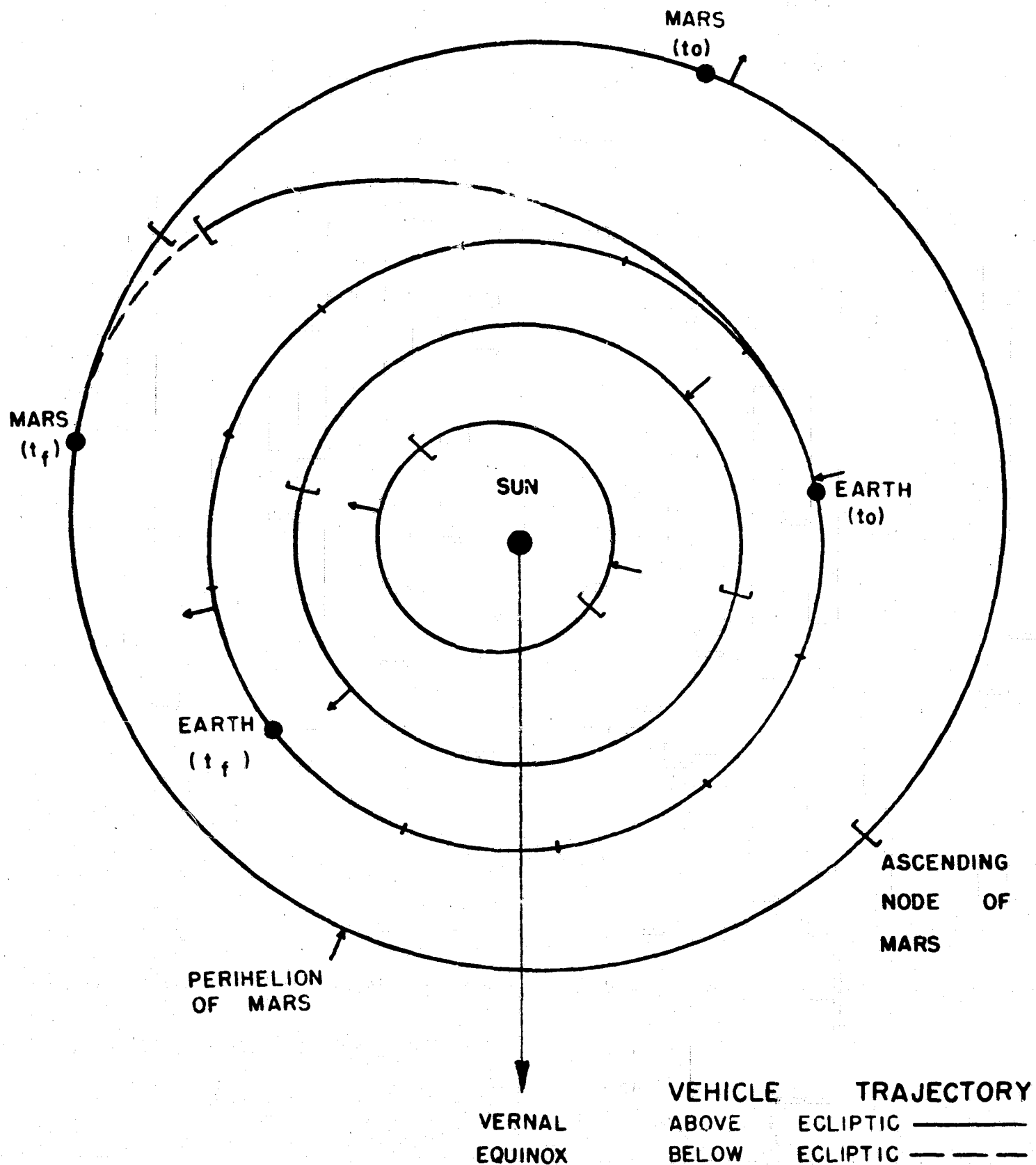
TRANSFER TIME : 268 DAYS



# FIG. 10 OPTIMUM RENDEZVOUS TRAJECTORY

DEPARTURE DATE : JAN 1 1967

TRANSFER TIME : 210 DAYS



# FIG. 11 OPTIMUM THRUST STEERING PROGRAM

DEPARTURE DATE : JAN 1 1967

TRANSFER TIME : 210 DAYS

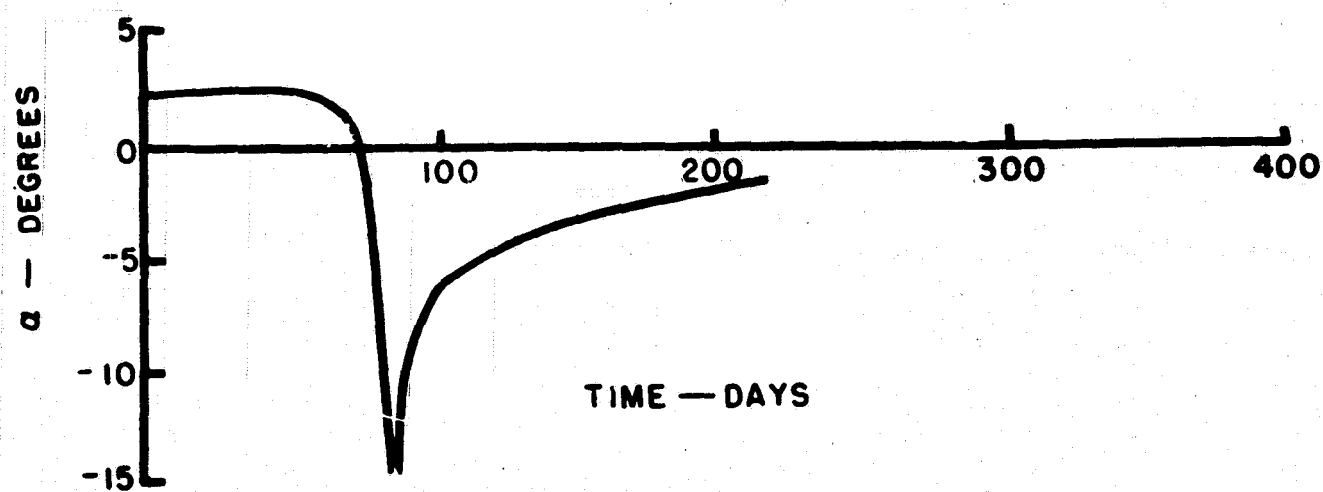
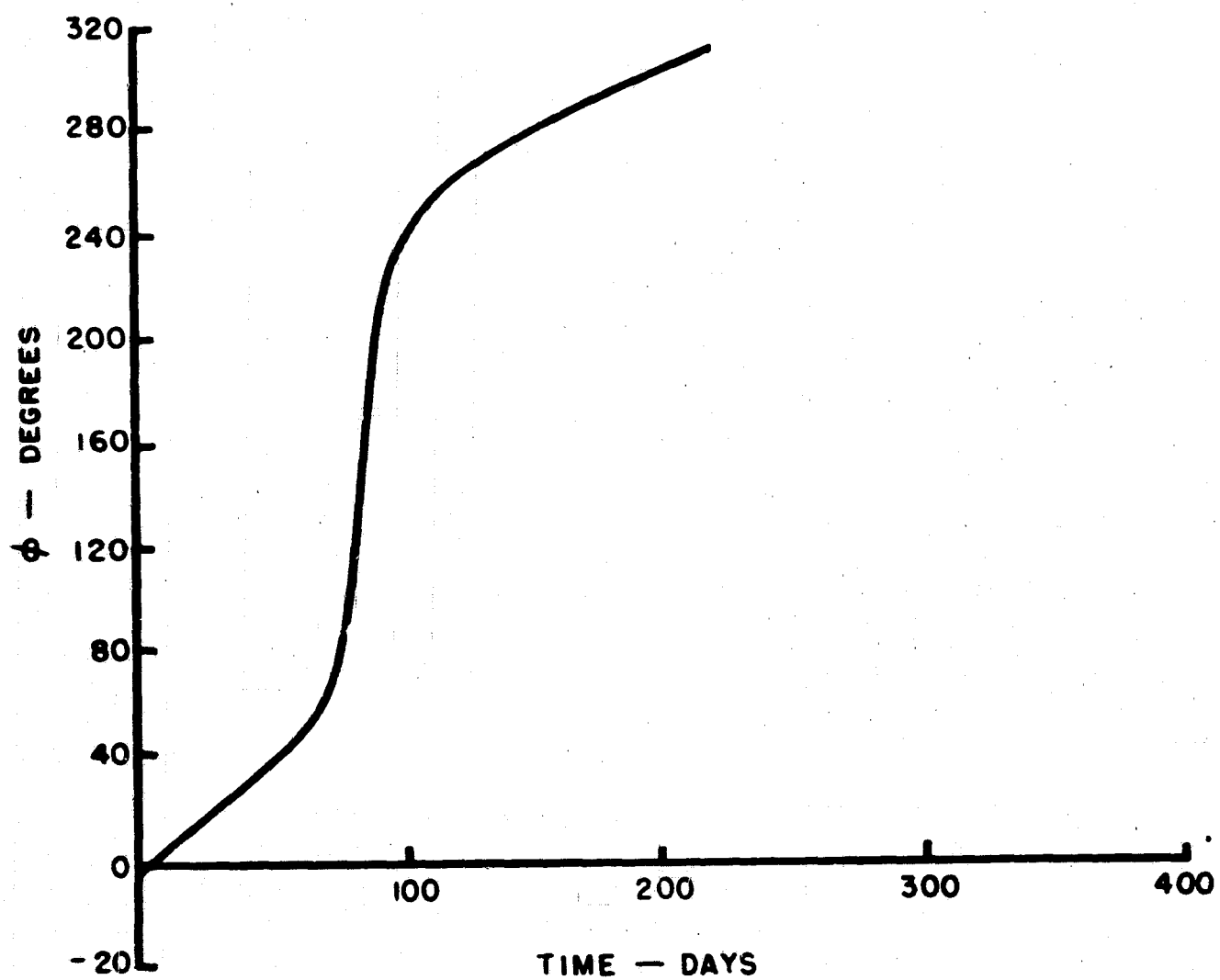
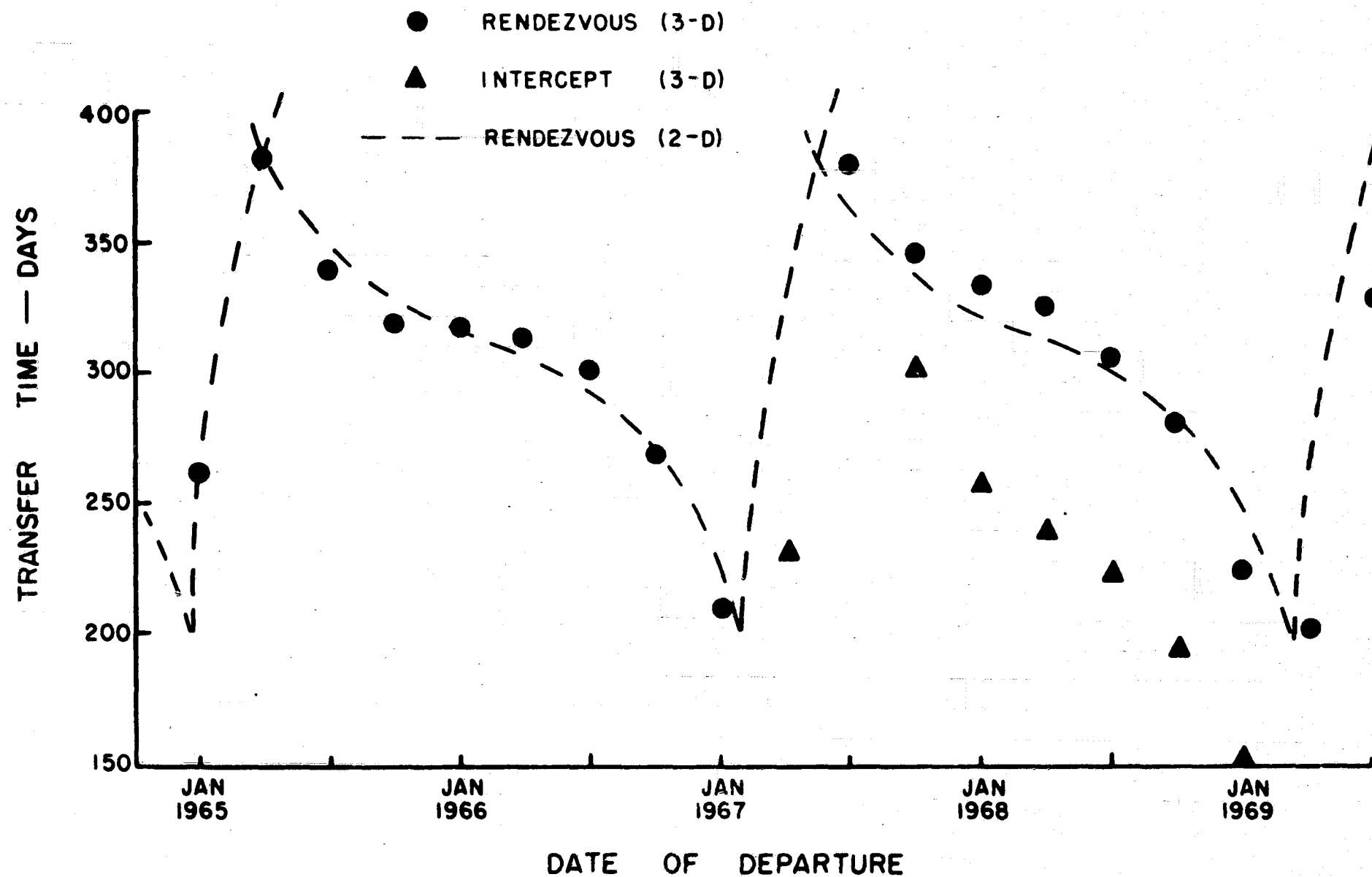


FIG. 12 TRANSFER TIMES FOR OPTIMUM RENDEZVOUS  
AND INTERCEPT TRAJECTORIES — THRUST CONSTANT



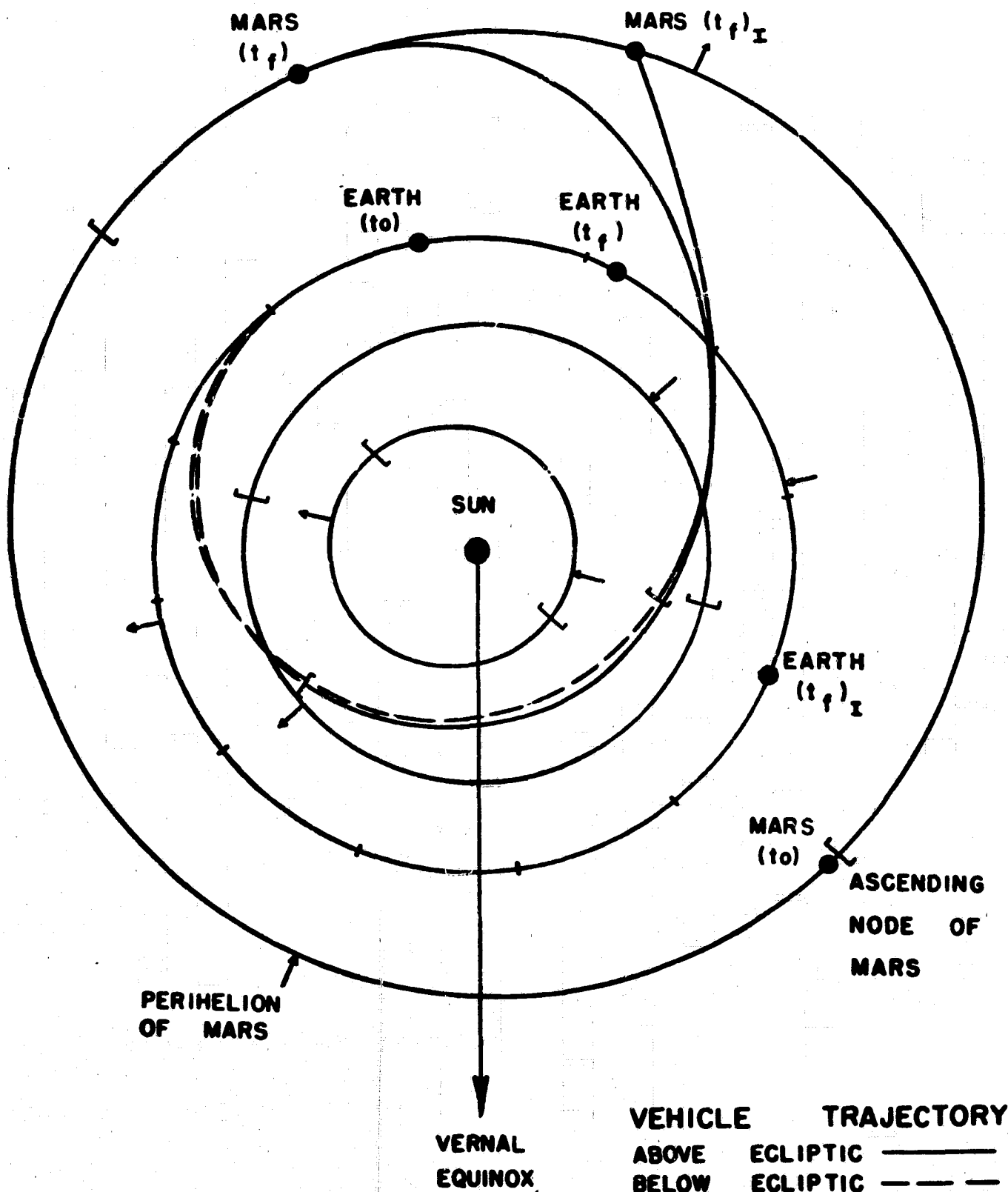
# FIG. 13 COMPARISON OF OPTIMUM RENDEZVOUS AND OPTIMUM INTERCEPT TRAJECTORIES

43

DEPARTURE DATE : APR 1 1968

TRANSFER TIMES : RENDEZVOUS: 326 DAYS

INTERCEPT: 240 DAYS



**RESEARCH DEPARTMENT  
GRUMMAN AIRCRAFT ENGINEERING CORPORATION**

**A REFERENCE SOLUTION FOR  
LOW-THRUST TRAJECTORIES**

By

**Gordon Pinkham**

**BETHPAGE, NEW YORK**

## SYMBOLS

$a$	arbitrary constant determining the initial magnitude of $f$ and $g$
$e$	base for natural logarithms
$\bar{e}$	eccentricity of a two-body orbit
$f, f'$	radial components of perturbing acceleration
$g, g'$	circumferential components of perturbing acceleration
$K$	gravitational constant of central body
$l, m, n$	constants associated with the perturbed solution
$p$	semilatus rectum of a two-body orbit
$q$	$\bar{e} \cos \omega$
$r$	radius from central body
$s$	$\bar{e} \sin \omega$
$t$	time
$u$	radial component of velocity, positive in the direction of increasing $r$
$v$	circumferential component of velocity, positive in the direction of motion
$V$	total velocity magnitude
$\theta$	angle in the plane of the trajectory, measured from a fixed line and positive in the direction of motion
$\omega$	angle of pericenter of a two-body orbit

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A REFERENCE SOLUTION FOR  
LOW-THRUST TRAJECTORIES

by

Gordon Pinkham

1969

Summary

This report presents an analytical solution to the trajectory equations for a low-thrust rocket when the thrust is tangential and varies nearly inversely with the square of the radius from the central body. The variation in the thrust magnitude has been so chosen that the solution possesses as many arbitrary constants as the order of the system of differential equations governing the motion. Thus an Encke or variation-of-parameters analysis of neighboring trajectories is practicable.

INTRODUCTION

The original impetus for the work described in this paper was a desire to write a variation-of-parameters or Encke scheme for integrating two-dimensional geocentric low-thrust trajectories. This trajectory problem presents many difficulties because of the large number of revolutions involved and the unacceptable accumulation of errors encountered with straightforward integration methods. A more sophisticated integration routine based on a spiral trajectory is an attractive device for eliminating some of

these problems. But the reference spiral must be expressed analytically to be of use, and at present only a very few are known. Among these are the logarithmic spiral and the solution for constant radial thrust. Both proved unsuitable after a preliminary examination — the first because it is not a general solution to the trajectory equations and the second because it does not closely approximate the trajectories we wish to integrate. A literature search failed to uncover any other usable solution, so an attempt was made to integrate the equations of motion with a suitable thrust program. Two such thrust programs seemed particularly interesting — constant circumferential thrust and constant tangential thrust; but every attempt to integrate them failed. It was noted, however, that the equations took on a very convenient and symmetric form when the thrust was assumed tangential; and allowing the thrust to vary in magnitude made them integrable. The resulting solution is more satisfactory in many ways than the logarithmic spiral. It is a general solution — possessing a sufficient number of arbitrary constants; and it reaches escape speed after a finite number of revolutions — something the logarithmic spiral never does. In addition, it is easily expressed as a function of sines, cosines, and exponentials. The following pages describe this solution, and a variation-of-parameters scheme based upon it is partially derived.

#### DERIVATION OF SOLUTION

The basic equations governing the planar motion of a body in a perturbed central force field are:

$$\begin{aligned}
 \frac{dr}{dt} &= u \\
 \frac{du}{dt} &= \frac{v^2}{r} - \frac{K}{r^2} + f \\
 \frac{dv}{dt} &= -\frac{uv}{r} + g \\
 \frac{d\theta}{dt} &= \frac{v}{r}
 \end{aligned}
 \tag{1}$$

In Eqs. (1)  $r$  is the radius;  $u$  and  $v$  are the radial and circumferential components, respectively, of the velocity;  $\theta$  is the angle in the plane of the orbit;  $f$  and  $g$  are the radial and circumferential components of the perturbing acceleration, and  $K$  is the gravitational constant of the central body. When  $f$  and  $g$  are zero the solution is a conic section expressible as a function of  $p$ ,  $q$ ,  $s$ , and  $\theta$  where  $p$  is the semilatus rectum,  $q = \bar{e} \cos \omega$ ,  $s = \bar{e} \sin \omega$ ,  $\bar{e}$  is the eccentricity, and  $\omega$  is the angle of pericenter. The derivation of our perturbed solution proceeds from the variation-of-parameters equations for the quantities  $p$ ,  $q$ , and  $s$ . These parameters may be expressed as functions of the radius and velocity components in the following manner:

$$p = \frac{r^2 v^2}{K}$$

$$q = \left( \frac{rv^2}{K} - 1 \right) \cos \theta + \frac{ruv}{K} \sin \theta \quad (2)$$

$$s = \left( \frac{rv^2}{K} - 1 \right) \sin \theta - \frac{ruv}{K} \cos \theta$$

Differentiating Eqs. (2) with respect to the time and substituting Eqs. (1) on the right, we derive the time derivatives of  $p$ ,  $q$ , and  $s$ . Multiplying by  $r/v$ , we change the independent variable to  $\theta$ , and we have

$$\frac{dp}{d\theta} = \left( 2 \frac{r^3}{K} \right) g$$

$$\frac{dq}{d\theta} = \left( \frac{r^2}{K} \sin \theta \right) f + \left( \frac{r^2 u}{Kv} \sin \theta + \frac{2r^2}{K} \cos \theta \right) g \quad (3)$$

$$\frac{ds}{d\theta} = \left( \frac{2r^2}{K} \sin \theta - \frac{r^2 u}{Kv} \cos \theta \right) g - \left( \frac{r^2}{K} \cos \theta \right) f .$$

Substituting the expressions for  $r$ ,  $u$ , and  $v$  as functions of  $p$ ,  $q$ ,  $s$ , and  $\theta$ , the right-hand sides of

Eqs. (3) become functions solely of  $p$ ,  $q$ ,  $s$ , and  $\theta$ . If we now set

$$\begin{aligned} f &= a \sqrt{Kp} u / 2r^2 \\ g &= a \sqrt{Kp} v / 2r^2, \end{aligned} \quad (4)$$

where  $a$  is an arbitrary constant, then the perturbing force is tangential and varies as  $\sqrt{p}/r^2$  and the following differential equations result:

$$\begin{aligned} \frac{dp}{d\theta} &= ap \\ \frac{dq}{d\theta} &= a(q + \cos \theta) \\ \frac{ds}{d\theta} &= a(s + \sin \theta) \end{aligned} \quad (5)$$

These equations have the solution

$$\begin{aligned} p &= l e^{a\theta} \\ q &= \frac{a}{a^2 + 1} (\sin \theta - a \cos \theta) + m e^{a\theta} \\ s &= \frac{-a}{a^2 + 1} (\cos \theta + a \sin \theta) + n e^{a\theta} \end{aligned} \quad (6)$$

The quantities  $l$ ,  $m$ ,  $n$  are constants to be determined by the initial conditions, and it should be noted that there are as many of these as there are equations in Eqs. (5). As mentioned previously, when Eqs. (6) are substituted into the expressions for  $r$ ,  $u$ , and  $v$ , a trajectory results which does not revolve indefinitely but reaches hyperbolic

velocity after a number of revolutions, the number depending on the magnitude of  $a$ . The expression for the radius is:

$$r = \ell e^{a\theta} / \left\{ \frac{1}{a^2 + 1} + e^{a\theta} (m \cos \theta + n \sin \theta) \right\} \quad (7)$$

The denominator has many zeros, but only the first has physical significance. This occurs when  $e^{a\theta} \sqrt{m^2 + n^2}$  approaches the magnitude of  $1/(a^2 + 1)$ ; therefore in this region the radius becomes infinite. This behavior might have been conjectured from the fact that the assumed perturbing force is larger at pericenter and smaller at apocenter than a thrust which varies simply as  $1/r^2$ .

With the solution of Eqs. (5) now available to us, we may seek information on those problems in which the perturbing forces differ by only a small amount from those assumed in Eqs. (4). To this end we derive the equations for the derivatives of the parameters  $\ell$ ,  $m$ , and  $n$  with respect to  $\theta$ . The derivation proceeds exactly as for the time derivatives of  $p$ ,  $q$ , and  $s$ ; so there is no need to repeat the steps here. The resulting equations are similar to Eqs. (3).

$$\frac{d\ell}{d\theta} = \left( 2 \frac{r^3}{K} \right) g' e^{-a\theta}$$

$$\frac{dm}{d\theta} = \left( \frac{r^2}{K} \sin \theta \right) f' e^{-a\theta} + \left( \frac{r^2 u}{Kv} \sin \theta + \frac{2r^2}{K} \cos \theta \right) g' e^{-a\theta} \quad (8)$$

$$\frac{dn}{d\theta} = \left( \frac{2r^2}{K} \sin \theta - \frac{r^2 u}{Kv} \cos \theta \right) g' e^{-a\theta} - \left( \frac{r^2}{K} \cos \theta \right) f' e^{-a\theta}$$

Here we have set

$$\begin{aligned} f' &= f - a(\sqrt{Kp} u/2r^2) \\ g' &= g - a(\sqrt{Kp} v/2r^2) \end{aligned} \quad (9)$$

If  $f'$  and  $g'$  are very small, Eqs. (8) should prove useful in integrating the trajectory numerically.

No attempt has been made here to integrate the time as a function of  $\theta$  or vice versa; and the indications are that this is a formidable task. Equations (8) may be used, nevertheless, in a variation-of-parameters scheme by changing the independent variable to the time and adding the differential equation for  $\theta$ . A comparison of this scheme with another based on a two-body orbit is planned. The variable  $\theta$  is not, of course, a parameter of the undisturbed motion; but experience indicates that in most applications it is a sufficiently smooth function of time so that its integration does not slow the computation of a trajectory. Moreover, the iterative solution of Kepler's equation or one like it is avoided.

It is hoped that the solution, Eqs. (6), may be of interest to some and that Eqs. (8) will have value in approximating or computing low-thrust trajectories.

**RESEARCH DEPARTMENT  
GRUMMAN AIRCRAFT ENGINEERING CORPORATION**

**OPTIMAL LOW-THRUST NEAR-CIRCULAR ORBITAL TRANSFER**

**By**

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18692 Summary

Optimum low-thrust transfers between neighboring circular orbits have been studied for a simplified system model. Small deviations from an original circular orbit motion are assumed and the thrust acceleration is taken to be constant. Within the limits of these assumptions, the numerical results are sufficient to describe in general the optimum thrust steering program and performance capability of a vehicle in terms of its thrust/weight ratio, orbital frequency, and thrust duration. For values of thrust duration equal to an integral multiple of the orbital period, the optimum thrust direction is continuously circumferential.

INTRODUCTION

In order to better understand the problems which may be encountered in geocentric low-thrust maneuvers and to gain insight into the character of optimal maneuvers, an analysis for a simplified system model has been carried out.

The equations of motion employed for the analysis are expressed in coordinates of a rotating axis system, with the origin moving in a circular orbit about the earth. By assuming that the vehicle's thrust/mass ratio is constant, and that the motion is always near this origin, the equations become linear with constant coefficients.

This system model is particularly suited to the geocentric low-thrust case because the relatively large gravitational forces prevent the vehicle from deviating significantly from its original circular orbit motion, even after many revolutions.

### Equations of Motion

The equations of motion are those employed by Clohessy and Wiltshire, in a satellite rendezvous analysis (Ref. 1). The motion is referred to a rotating axis system whose origin moves in a circular orbit about the earth, as shown in Fig. 1. The positive  $y$ -axis points away from the gravitational center and the  $x$ -axis is oriented along the tangent to the circular orbit with the positive direction opposite to that of the orbital velocity vector. Only the planar motion case is analyzed. If the displacement of the vehicle from the origin is assumed small, the equations simplify to the following on linearization:

$$m\ddot{x} = T \cos \psi + 2m\omega\dot{y}$$

$$m\ddot{y} = T \sin \psi - 2m\omega\dot{x} + 3m\omega^2 y$$

Here  $T$  is the thrust,  $m$  is the mass of the vehicle, and  $\omega$  is the angular frequency of the origin's circular orbit motion. Dots indicate differentiation with respect to time.

These equations simplify further by transformation to a new independent variable,

$$\tau = \omega t$$

The variable  $\tau$  is nondimensional and may be interpreted physically as the orbital angle between the rotating origin and some inertial reference (Fig. 1). The equations of motion may be written in first order form as:

$$u' = P \cos \psi + 2v \quad (1)$$

$$v' = P \sin \psi - 2u + 3y \quad (2)$$

$$x' = u \quad (3)$$

$$y' = v \quad (4)$$

where the prime indicates differentiation with respect to the new variable  $\tau$ , and

$$P = \frac{T}{m\omega^2}$$

If  $P$  is assumed constant and the thrust steering angle,  $\psi$ , is a function of  $\tau$  only, then the solution of the system (1) to (4) is

$$u(\tau_1) = 2[2u_0 - 3y_0]\cos \tau_1 + 2v_0 \sin \tau_1 + 6y_0$$

$$- 3u_0 + P \int_{\tau_0}^{\tau_1} \left\{ [4 \cos (\tau_1 - \tau) - 3] \cos \psi \right.$$

$$\left. + 2 \sin(\tau_1 - \tau) \sin \psi \right\} d\tau$$

$$v(\tau_1) = v_0 \cos \tau_1 + [3y_0 - 2u_0] \sin \tau_1$$

$$+ P \int_0^{\tau_1} \left\{ -2 \sin(\tau_1 - \tau) \cos \theta \right.$$

$$\left. + \cos(\tau_1 - \tau) \sin \theta \right\} d\tau$$

$$x(\tau_1) = 2(2u_0 - 3y_0) \sin \tau_1 - 2v_0 \cos \tau_1$$

$$+ (6y_0 - 3u_0) \tau_1 + (x_0 + 2v_0)$$

$$+ P \int_0^{\tau_1} \left\{ [4 \sin(\tau_1 - \tau) - 3(\tau_1 - \tau)] \cos \theta \right.$$

$$\left. + 2[1 - \cos(\tau_1 - \tau)] \sin \theta \right\} d\tau$$

$$y(\tau_1) = (2u_0 - 3y_0)\cos \tau_1 + v_0 \sin \tau_1 + 4y_0$$

$$- 2u_0 + P \int_0^{\tau_1} \left\{ 2[\cos(\tau_1 - \tau) - 1]\cos \theta \right. \\ \left. + \sin(\tau_1 - \tau)\sin \theta \right\} d\tau$$

where  $\tau_1$  is arbitrary and  $u_0, v_0, x_0, y_0$  are initial conditions. The free-fall ( $P=0$ ) part of the above solution was first introduced by Wheelon (Ref. 2) and Anthony (Ref. 1).

Prior to maneuvering, the vehicle is considered to be in a circular orbit and coincident with the origin of the reference system. Hence, all initial conditions are zero. The equations for the final velocity and position components reduce to

$$u_f = P \int_0^{\tau_f} \left\{ [4 \cos(\tau_f - \tau) - 3]\cos \theta \right. \\ \left. + 2 \sin(\tau_f - \tau)\sin \theta \right\} d\tau \quad (5)$$

$$v_f = P \int_0^{\tau_f} \left\{ -2 \sin(\tau_f - \tau) \cos \theta + \cos(\tau_f - \tau) \sin \theta \right\} d\tau \quad (6)$$

$$x_f = P \int_0^{\tau_f} \left\{ [4 \sin(\tau_f - \tau) - 3(\tau_f - \tau)] \cos \theta + 2[1 - \cos(\tau_f - \tau)] \sin \theta \right\} d\tau \quad (7)$$

$$y_f = P \int_0^{\tau_f} \left\{ 2[\cos(\tau_f - \tau) - 1] \cos \theta + \sin(\tau_f - \tau) \sin \theta \right\} d\tau \quad (8)$$

### Variational Treatment

A class of optimal maneuvers of considerable generality is that in which some function of the terminal values  $u_f$ ,  $v_f$ ,  $y_f$ ,  $x_f$  and the terminal value of the time parameter is a minimum. If Lagrange multipliers  $\Lambda_1$ , ...,  $\Lambda_4$  are introduced in the usual way, the vanishing of the first variation of the expression

$$J = \Lambda_1 u_f + \Lambda_2 v_f + \Lambda_3 x_f + \Lambda_4 y_f$$

provides a necessary condition for a minimum. In the present problem the multipliers employed are constants since the terminal values of the state variables are represented in terms of definite integrals. The multipliers  $\Lambda_i$  are the terminal values of the multiplier functions  $\lambda_i(t)$  which would be introduced in a treatment based upon the differential equations (1)-(4).

If  $F$  is the (collected) integrand of the integrals appearing in  $J$

$$F = \Lambda_1 a_1 \cos \theta + \Lambda_1 b_1 \sin \theta + \Lambda_2 a_2 \cos \theta + \Lambda_2 b_2 \sin \theta \\ + \Lambda_3 a_3 \cos \theta + \Lambda_3 b_3 \sin \theta + \Lambda_4 a_4 \cos \theta + \Lambda_4 b_4 \sin \theta ,$$

where

$$\begin{aligned} a_1 &= 4 \cos(\tau_f - \tau) - 3 & b_1 &= 2 \sin(\tau_f - \tau) \\ a_2 &= -2 \sin(\tau_f - \tau) & b_2 &= \cos(\tau_f - \tau) \\ a_3 &= 4 \sin(\tau_f - \tau) - 3(\tau_f - \tau) & b_3 &= 2 - 2 \cos(\tau_f - \tau) \\ a_4 &= 2 \cos(\tau_f - \tau) - 2 & b_4 &= \sin(\tau_f - \tau) , \end{aligned}$$

the vanishing of the first variation of  $J$  implies

$$\frac{\partial F}{\partial \theta} = 0$$

which leads to

$$\tan \vartheta = \frac{A}{B}$$

where

$$A = \Lambda_1 b_1 + \Lambda_2 b_2 + \Lambda_3 b_3 + \Lambda_4 b_4$$

$$B = \Lambda_1 a_1 + \Lambda_2 a_2 + \Lambda_3 a_3 + \Lambda_4 a_4$$

The Legendre-Clebsch condition

$$\frac{\partial^2 F}{\partial u^2} \geq 0$$

resolves the ambiguity of quadrant for the angle  $\vartheta$ , yielding

$$\sin \vartheta = \frac{-A}{\sqrt{A^2 + B^2}} \quad \cos \vartheta = \frac{-B}{\sqrt{A^2 + B^2}}$$

We wish to study transfers between neighboring circular orbits which are optimal in the sense of maximizing the terminal values of  $y$ , the increase in radius. Hence

$$\Lambda_4 = -1$$

Also, because no specification is to be imposed upon the final value of  $x_f$  (unlike the case of rendezvous maneuvers)

$$\Lambda_3 = 0$$

With these values of  $\Lambda_3$  and  $\Lambda_4$  the optimum thrust steering formula, including resolution of quadrant ambiguity (Legendre-Clebsch condition), reduces to

$$\tan \theta = \frac{(1-2\Lambda_1)\sin(\tau_f - \tau) - \Lambda_2 \cos(\tau_f - \tau)}{2\Lambda_2 \sin(\tau_f - \tau) + 2(1-2\Lambda_1)\cos(\tau_f - \tau) + 3\Lambda_1 - 2} \quad (9)$$

$$\sin \theta = \frac{C}{\sqrt{C^2 + D^2}} \quad \cos \theta = \frac{D}{\sqrt{C^2 + D^2}}$$

where

$$C = (1 - 2\Lambda_1)\sin(\tau_f - \tau) - \Lambda_2 \cos(\tau_f - \tau)$$

$$D = 2\Lambda_2 \sin(\tau_f - \tau) + 2(1 - 2\Lambda_1)\cos(\tau_f - \tau) + (3\Lambda_1 - 2)$$

Values of the two remaining constants  $\Lambda_1$  and  $\Lambda_2$  are sought for which the terminal velocity components satisfy circular orbit conditions. For the present linear analysis the final velocity components should be

$$u_f = 3y_f/2 \quad (10)$$

$$v_f = 0 \quad (11)$$

### Computer Program

The IBM 7090 computer was programmed to integrate numerically the equations of motion, generating solutions for prescribed values of  $\Lambda_1$ ,  $\Lambda_2$ , and  $\tau_f$ . A systematic survey was made by varying  $\Lambda_1$  and  $\Lambda_2$ , for a given value of  $\tau_f$ , until the terminal constraints were satisfied.

The computer program was modified to automatically "home-in" on those constants. To repeat this procedure for other values of  $\tau_f$  would have been uneconomical and hence a perturbation scheme was employed to generate solutions for each succeeding value of  $\tau_f$  by extrapolating the results of previously iterated solutions. Newton's formula for backward interpolation (Ref. 3) is ideally suited to this purpose and becomes quite simple for constant  $\tau_f$  intervals.

### Computational Results

Seventy-two iterated solutions covering four orbital periods were computed on the IBM 7090 in less than twenty-five minutes. Figs. 2 through 8 summarize these results. An examination of the optimum orbit transfer maneuvers reveals several interesting characteristics.

It is noted (Fig. 2) that the time variation of the thrust steering angle is antisymmetrical with respect to the midpoint. Similarly, time histories of the two components of velocity and acceleration also have this "exactly" symmetrical or antisymmetrical property. "Exactly" refers to an accuracy of six significant figures for the numerically integrated results, and suggests that the symmetry property would be exact if the integration were error-free.

Figure 3 shows the trajectories of the vehicle with respect to the rotating axis system for thrust programs shown in Fig. 2.

Thrust steering angles for longer transfer times are shown in Fig. 4. Although the same antisymmetrical properties are displayed, it is of interest to note that the  $\psi$  motion has a mean of  $\psi = 180^\circ$  (circumferential thrust) whereas the corresponding motion for the short term maneuvers (Fig. 2) takes place about a mean of  $\psi = 0$  (opposite to circumferential thrust).

This abrupt change in character of the optimum steering program has been made more evident in Fig. 5 by rescal-

ing the time of each solution so that a comparison may be made on a common normalized time scale. When this is done, it becomes obvious that between  $\tau_f = \pi$  and  $\tau_f = 4\pi/3$  the time variation becomes more abrupt. A closer examination reveals that there exists a discontinuous solution at  $\tau_f = 1.2163\pi$  for which case  $\theta$  changes instantaneously from  $90^\circ$  to  $-90^\circ$  at the midpoint of the maneuver. The instantaneous change could also be interpreted to be from  $90^\circ$  to  $270^\circ$ . This has been the only discontinuous solution encountered for circle-to-circle orbital transfer.

Also shown in Figs. 4 and 5 is the thrust steering angle for  $\tau_f = 2\pi$ . For this solution, which corresponds to an orbital transfer of just one revolution's duration,  $\theta$  remains constant at  $180^\circ$ . In fact, whenever the duration of powered flight is some integral multiple of the orbital period, the optimum thrust direction is circumferential and the vehicle passes through a higher energy circular orbit condition at the end of each revolution. The possibility that the optimal thrust direction program may be circumferential for a circle-to-circle orbital transfer involving many revolutions should offer an exploitation possibility in connection with the guidance problem. In general, for the longer duration maneuvers,  $\theta$  oscillates at the orbital frequency. This is shown in Fig. 6 for maneuver durations of  $1\frac{1}{2}$ ,  $2\frac{1}{2}$ , and  $3\frac{1}{2}$  revolutions. It is interesting to note that, as this angle (or final nondimensional time) increases, the amplitude of the oscillation decreases and a steady-state circumferential thrust program is approached for maneuvers of very long duration.

Because the equations of motion are linear and contain only the single parameter  $P$ , it is possible to plot a "miles-per-gallon" parameter as a function of the nondimensional transfer time. This very general type of plot is presented in Fig. 7. Given the thrust/weight ratio of the vehicle and the frequency of the reference orbit, the increase in circular-orbit radius may be determined from Fig. 7 for any value of the orbital transfer angle.

It may be conjectured that the parameter  $P$  could be utilized as a basis of comparison for vehicles with different performance abilities and in dissimilar gravity environments, i.e., the ratio  $P$  is a measure of the relative influence of the thrust/force field.

### Further Analytical Investigation

With the insight gained, it is of interest to consider the possibility of analytical solutions.

If the three integral solutions (5), (6), (8) are substituted into the equations for terminal circular orbit requirements, (10) and (11), the following integrals must vanish:

$$\int_0^{\tau_f} [2 \cos(\tau_f - \tau) \cos \theta + \sin(\tau_f - \tau) \sin \theta] d\tau = 0 \quad (12)$$

$$\int_0^{\tau_f} [\cos(\tau_f - \tau) \sin \theta - 2 \sin(\tau_f - \tau) \cos \theta] d\tau = 0 \quad (13)$$

If the circumferential thrust program is inserted in (12) and (13), the integrals are easily evaluated and, indeed, both vanish as expected.

A fruitful result of the numerical study is the anti-symmetry property which implies that  $\tan \theta$  in (9) is zero at the midpoint in time. This leads to the explicit relationship between  $\Lambda_1$  and  $\Lambda_2$ :

$$\Lambda_2 = (1 - 2\Lambda_1)\tan(\tau_f/2)$$

The problem now reduces to a search for a one-parameter family of solutions which must be made to satisfy circular orbit terminal conditions. The resulting equations and integrals to be evaluated are still quite formidable, and as yet have not yielded to analytical treatment.

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FIG. 1 ORBIT TRANSFER SCHEMATIC FOR  
LINEAR LOW-THRUST ANALYSIS

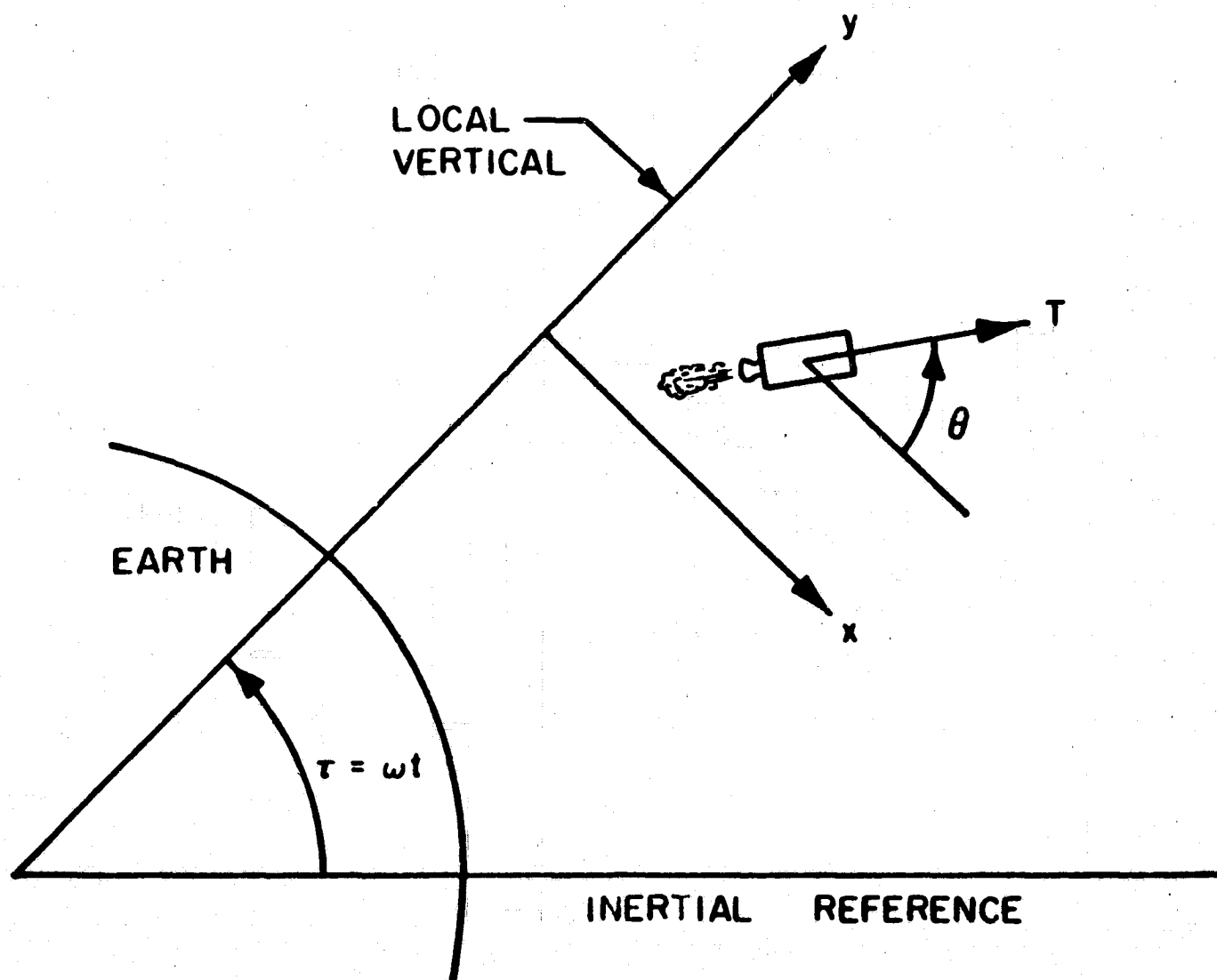


FIG. 2 THRUST DIRECTION TIME HISTORIES FOR OPTIMUM  
CIRCLE-TO-CIRCLE ORBITAL TRANSFER

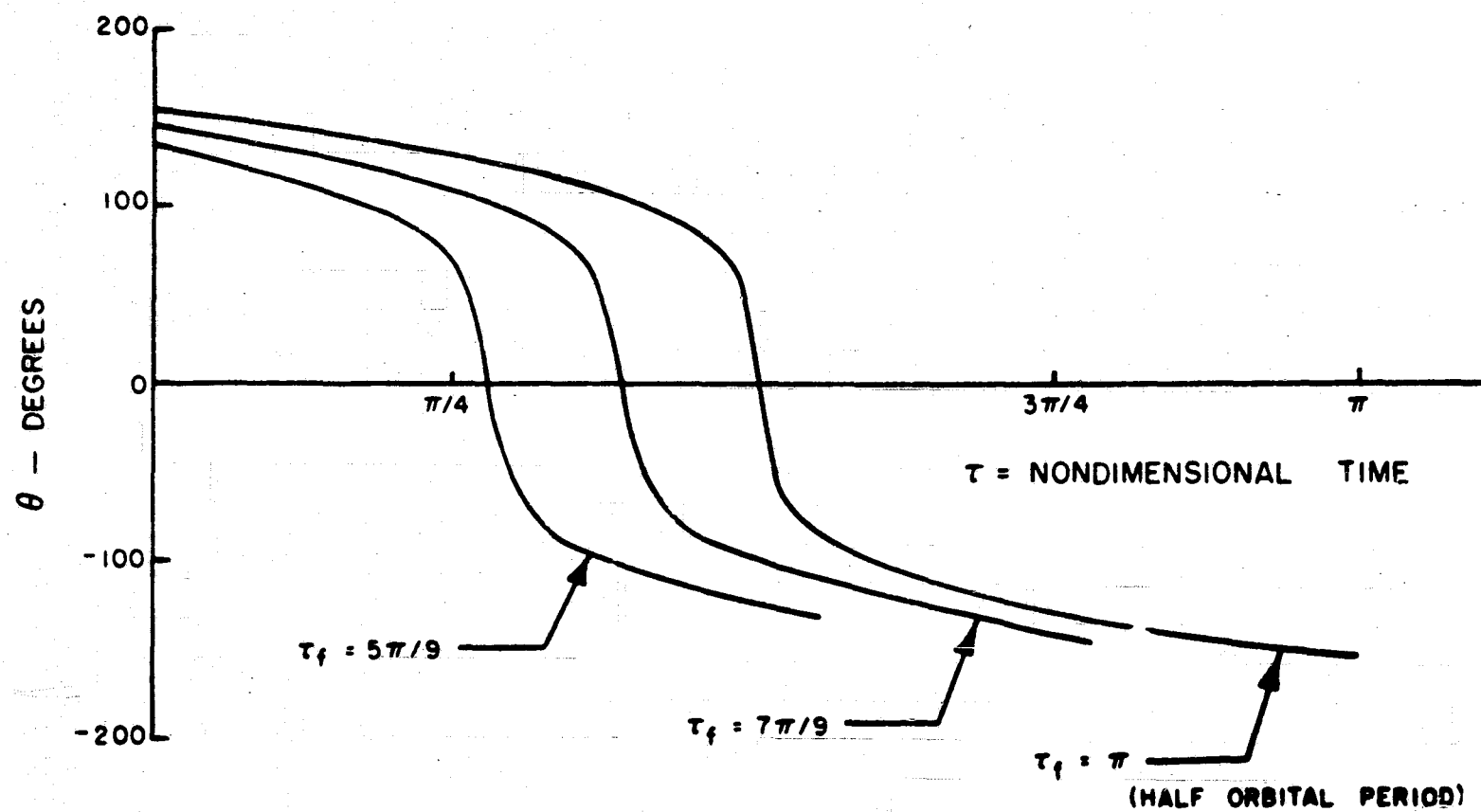


FIG. 3 TYPICAL TRAJECTORIES FOR OPTIMUM CIRCLE-TO-CIRCLE ORBITAL TRANSFER

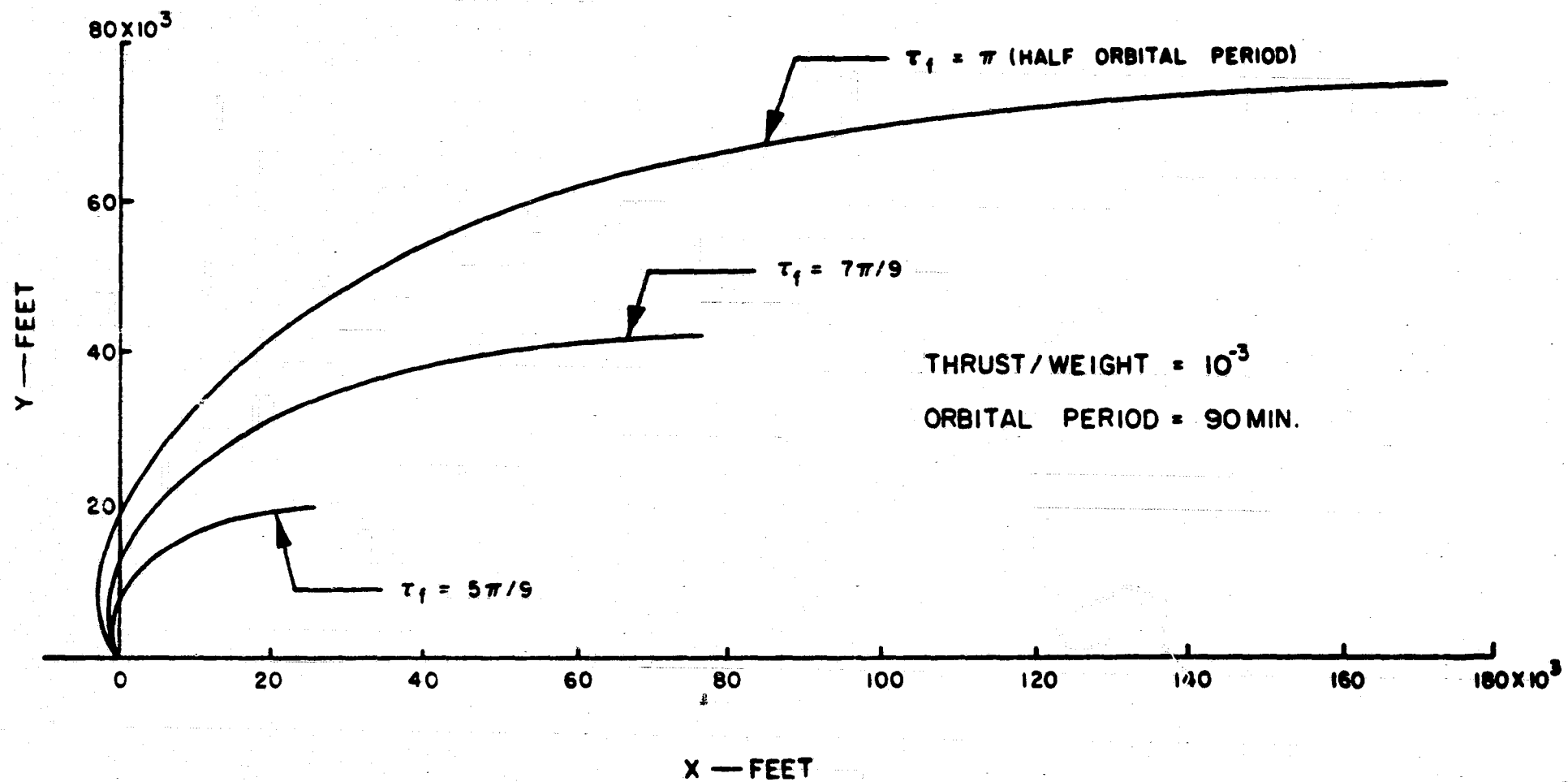


FIG. 4 THRUST DIRECTION TIME HISTORIES FOR OPTIMUM  
CIRCLE-TO-CIRCLE ORBITAL TRANSFER

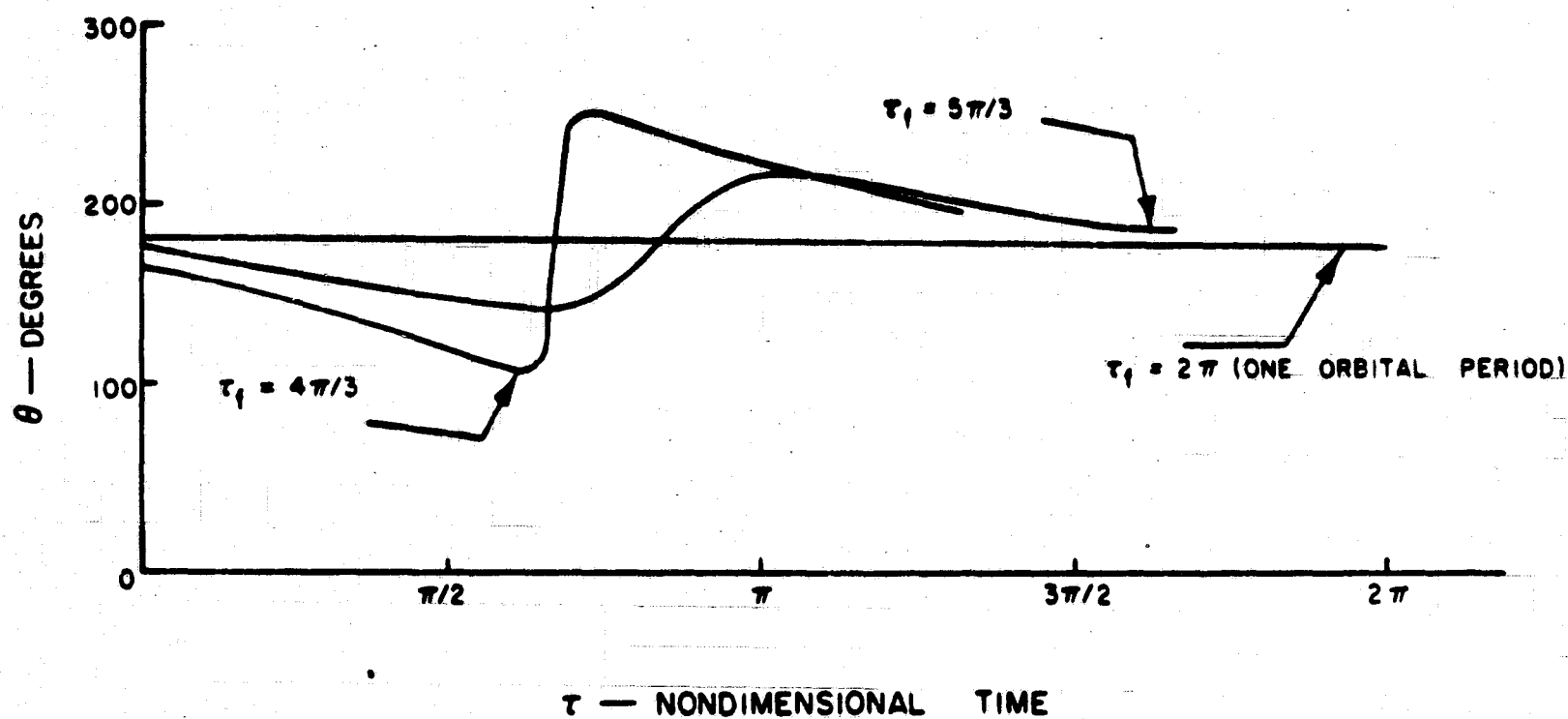


FIG. 5 NORMALIZED TIME HISTORIES OF THRUST  
DIRECTION ANGLE FOR TRANSFER TIMES  
UP TO ONE ORBITAL PERIOD

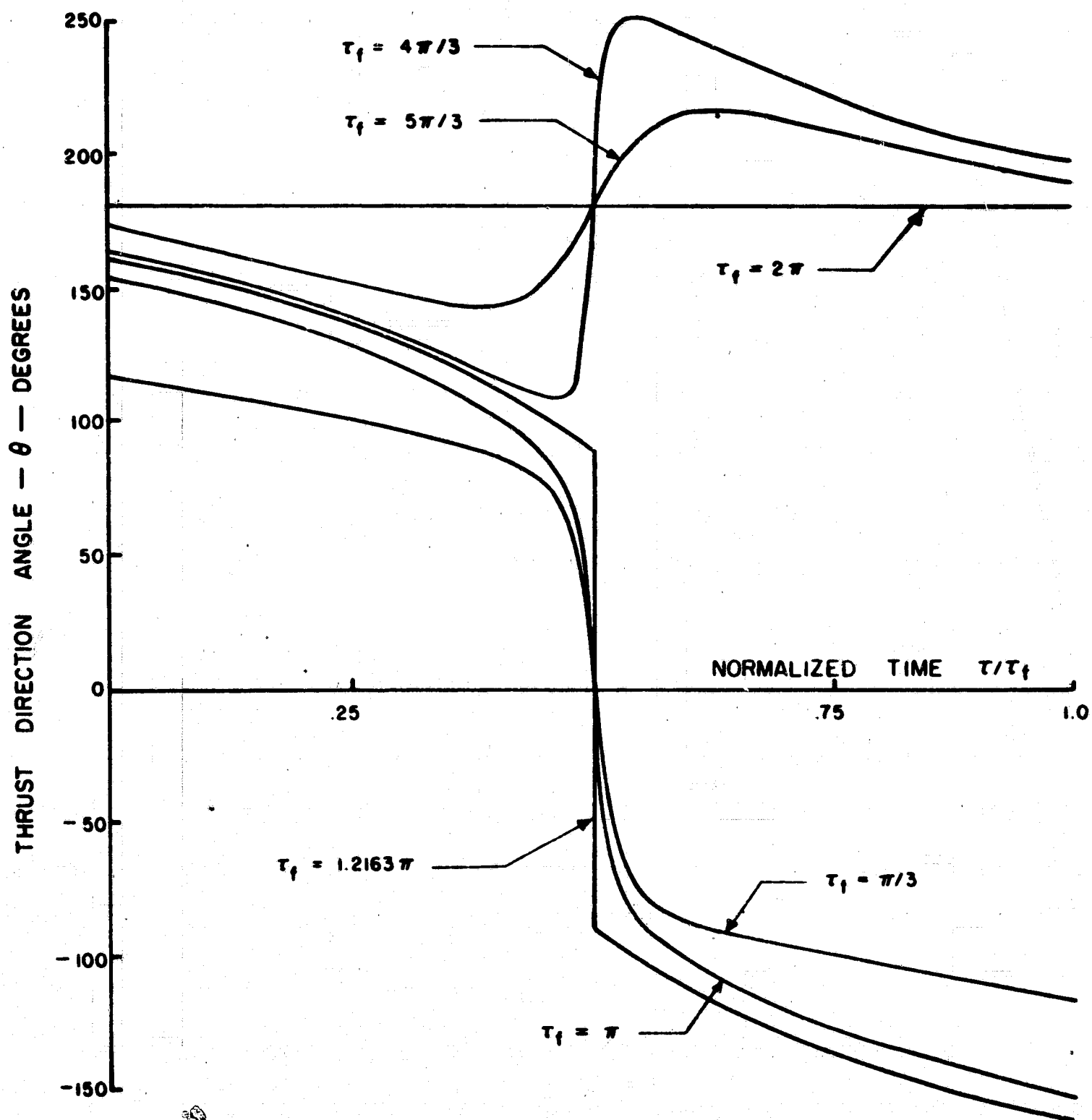


FIG. 6 TIME HISTORIES OF THRUST DIRECTION ANGLE FOR  
TRANSFER TIMES EXCEEDING ONE ORBITAL PERIOD

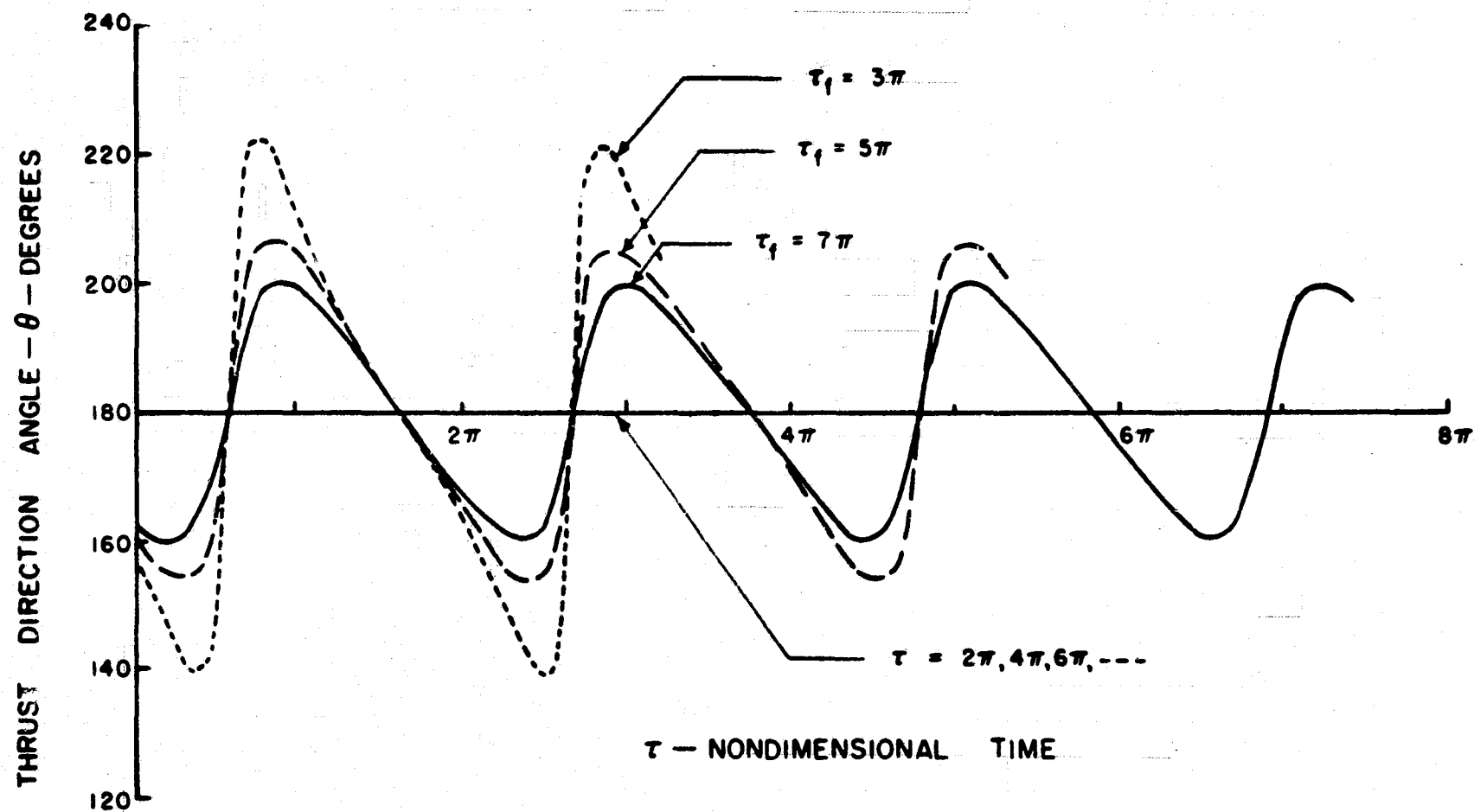


FIG. 7 VARIATION OF ALTITUDE GAIN PARAMETER WITH  
NONDIMENSIONAL TRANSFER TIME

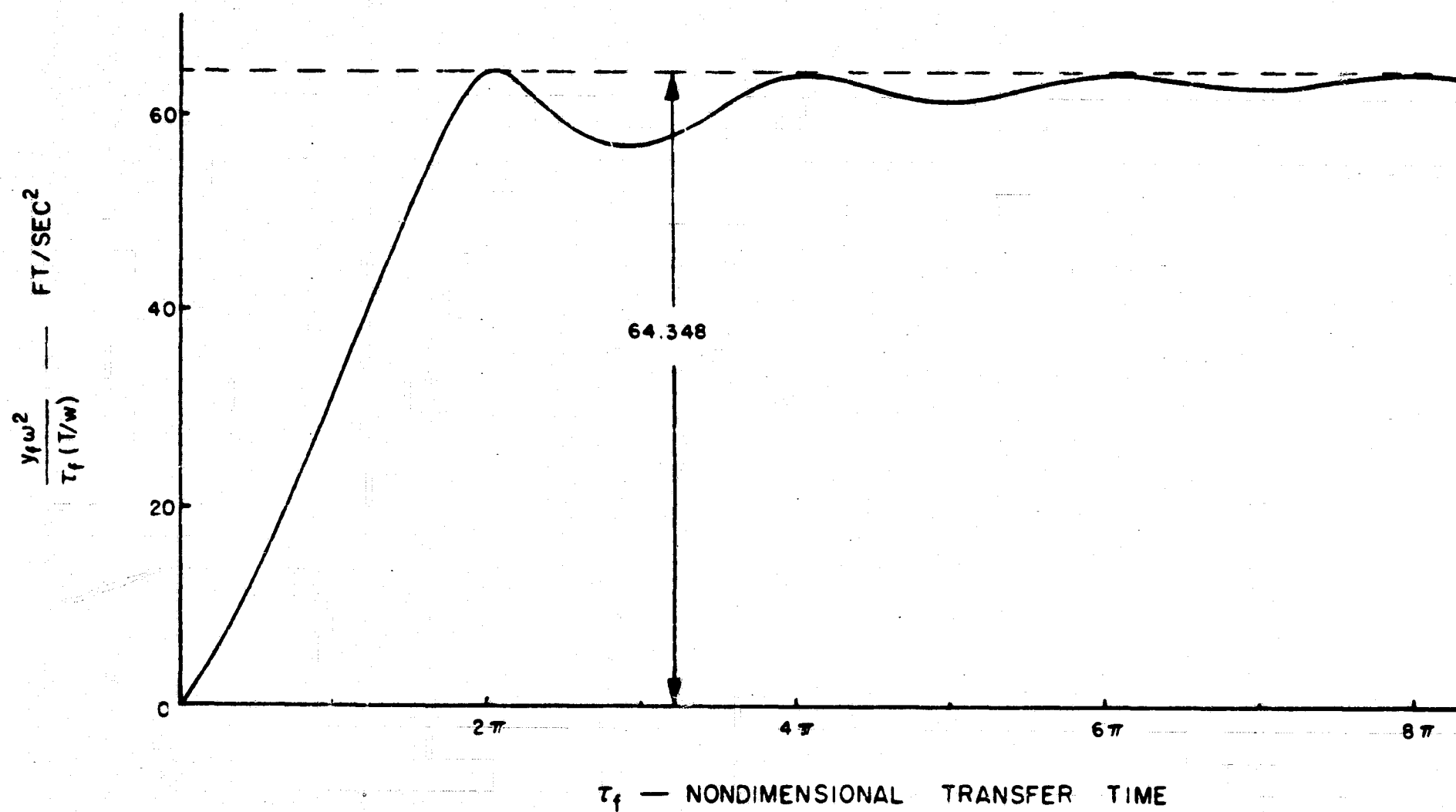
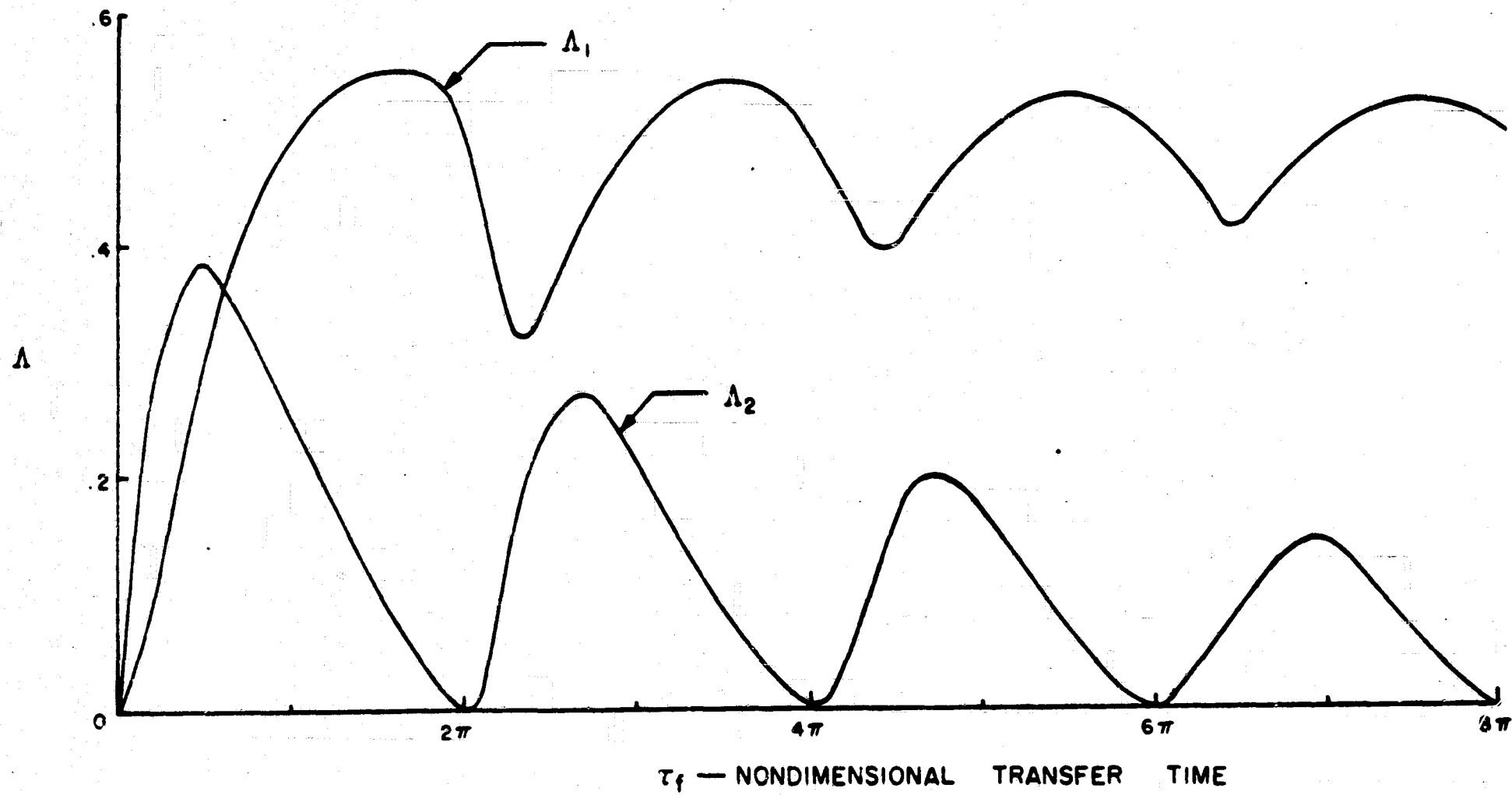


FIG. 8 VARIATION OF  $\Lambda_1$  AND  $\Lambda_2$  WITH  
NONDIMENSIONAL TRANSFER TIME



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**THE CALCULUS OF VARIATIONS APPROACH  
TO CONTROL OPTIMIZATION**

by

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## SUMMARY

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The present work is an attempt to present a general unified study of the first necessary conditions for an optimum trajectory with special emphasis on the geometrical interpretation of some of these conditions. In this study the pertinent necessary conditions are derived in a simple manner, although trying to preserve the necessary rigorousness of formulation. The introduction of the generalized canonical transformation presented is shown to be of importance in deriving the "Maximality Principle" and graphically interpreting this principle and other necessary conditions. The corner point conditions are also analyzed and their geometrical interpretation is included.

In this paper the author is therefore suggesting the application of interesting numerical-geometrical techniques of analysis for the treatment of complicated problems in trajectory optimization studies.

In the study presented here the derivations usually available in the literature on the Calculus of Variations (e.g., Refs. 1,2,4), for problems of the Lagrangian form, are now performed for a generalized Mayer formulation which accounts for the occurrence of control variables in the side conditions, the latter being assumed of a general form. The following comments on the contents and contributions given in this work are in order.

## INTRODUCTION

## A. SECTION I

In this section the original Mayer problem is expressed as one of the Bolza form and the first necessary conditions are therefore derived in a general manner obtaining the natural end-conditions and Euler equations from a single equation of variation. The Transversality Condition is given in different forms [Eqs. (10), (11), (12)]. In particular, its expression in terms of the constant Lagrange multipliers is of theoretical as well as practical interest in defining the normality conditions. This aspect, not previously presented in the literature on trajectory optimization, has also been introduced and applied by this author in Ref. 18.

The variational problem is formulated in non-parametric and parametric forms. For an admissible one-parameter family having corners located on a "line of corners" is shown that the parametric form offers the advantage of giving the complete set of Erdmann-Weierstrass vertex conditions in addition to the natural end-conditions (Transversality condition) and equations of variations (Euler equations). It is well to mention here that this formulation is the basis of the important Weierstrass's theory on parametric problems, developed originally by

Weierstrass in his lectures (1872), and considered of fundamental value for the study of geometrical variational problems.

Topics not usually found in the applied literature, as the conditions for non-singularity and normality of the extremals, are also considered.

The necessary Weierstrass condition is derived by considering arcs of three adjacent families properly defined in adjoining intervals. The Legendre condition, as generalized by Clebsch and Mayer, is obtained from a Taylor expansion of the Weierstrass condition in a close neighborhood of a point on the extremal.

## B. SECTION II

Here a generalized canonical transformation of the variational problem is introduced. Within the necessary rigorousness of formulation the concept of slopes and multipliers of a field is applied leading, through the application of the Legendre transformation of the variational problem, to the canonical differential equations of the extremals. The equivalence between the necessary conditions in canonical form and the necessary conditions derived in Section I is pointed out.

The analysis presented in this Section is essentially based on the original work of Hamilton and Lagrange where they introduce the canonical variables. More recent treatments by Bliss, Courant and Hilbert, and Pontryagin have also served as valuable reference. In this Section we have tried to extend previous work by including control variables in the constraints, applying the concept of a field and assuming a general form for the side conditions. Also the author has tried to present these topics following a didactically orderly derivation showing the correspondence of the results obtained with previous results of the Mayer problem. To this extent the work of Bliss in his book (Ref. 1) has undoubtedly been of relevant value.

## C. SECTION III

To the extent of introducing the case where bounded control variables are present in the side conditions, the problem with admissible one-sided control variations is here formulated. Applying the Mayer formulation of Section I some inequalities, not previously derived in the literature, are obtained as necessary conditions. The study follows similar lines as for the case of one-sided admissible variations in the state variables studied by Bliss and Underhill (Ref. 13) for problems of the Lagrangian form. In this area of studies it is of extreme value the work by Bliss on sufficient conditions in Ref. 27. He finds that with suitably strengthened inequalities as necessary conditions, the extremal in common with the boundary is a minimum (absolute minimum) when compared with any other extremal of the field having one-sided variations.

#### D. SECTION IV

The so-called Pontryagin's Maximum Principle is here obtained as the canonical expression of the Weierstrass condition. This is done by applying the general canonical transformation introduced in previous Sections and writing the Weierstrass expression in terms of canonical variables. The case of bounded control variables is treated by assuming one-sided variations. Thus, a set of necessary conditions in the form of inequalities and equivalent to those previously derived in Section III, are found. The inequalities obtained in this Section are expressed in terms of the Hamiltonian H-function instead of the Euler-Lagrange sum II as before. These developments are of theoretical importance in showing the perfect correspondence of the necessary conditions in both formulations.

Finally, some corner conditions, to be considered in addition to the Erdmann-Weierstrass corner continuity requirements, are discussed.

#### E. SECTION V

The equations of constraint are now specialized into a form which, in general, is analogous to that occurring in problems of the dynamics of flight. This is therefore a particular case of the general developments in previous Sections.

The Euler equations and the canonical differential equations of the extremals are derived applying the necessary conditions deduced in the preceding Sections.

The characteristic line and the H-line are analyzed and their properties, inasmuch as the graphical representation of the necessary conditions for an extremal is concerned, discussed.

The geometrical interpretation is based on the work originally performed by Zermelo, Caratheodory and more recently by Cicala (Ref. 3).

Here, several extensions to the geometrical interpretation of the necessary conditions are offered. In fact, the Weierstrass condition for cases of multiple control, time-dependent characteristics and H-lines, corners with continuous and discontinuous control, graphical representation of the conditions for admissible corner with control jump and the H-function along the extremal for the case where H does not contain the independent variable explicitly are some examples.

The case of problems with bounded control variables is also studied and geometrically interpreted using the characteristic line and the H-line.

## F. SECTION VI

The work is concluded by presenting some applications of interest in the study of discontinuous solutions. Our interest is in showing the geometrical interpretation of some necessary conditions, the form of the characteristic lines and H-lines, and the application of the necessary conditions discussed throughout the preceding sections. Here no emphasis is made on the physical problem or on its engineering implications. A more detailed analysis of these aspects is available in the references included.

## DEFINITION OF SYMBOLS

$\alpha$	Speed of sound (ft/sec.)
$C_{D_0}$	Zero-lift pressure drag coefficient
$C_D$	Drag coefficient
$C_{D_i} = K C_L^2$	Induced drag coefficient
$C$	Corner point
	Admissible comparison arc
$D$	Aerodynamic drag (lb.)
$= \frac{D}{mg}$	Non-dimensional drag.
$E$	Extremal arc
$F$	Euler-Lagrange sum in parametric form; field of extremals or enlarged integrand
$g$	Acceleration of gravity
$H$	Hamiltonian
$h$	Flight altitude (ft.)
$h = \frac{hg}{v_R^2}$	Non-dimensional altitude
$J$	Function to be minimized
$K = \frac{\partial C_{D_i}}{\partial (C_L^2)}$	Induced drag factor
$L$	Lifting force (lb.)
$= \frac{L}{mg}$	Non-dimensional lift
$M = \frac{v}{\alpha}$	Mach number

$q$	generalized coordinate
$t$	Time (sec.)
$v$	canonical variable
$V$	Flight velocity (ft/sec.)
$W$	Weierstrass condition
$x$	Range (ft.)
$Z = \frac{V}{V_R}$	Non-dimensional flight velocity
$\epsilon$	Parameter
$\xi = \frac{xg}{V_R^2}$	Non-dimensional range, (and indicates also admissible variation of the independent variable at the end-points).
$\eta$	Admissible variation of the generalized coordinate, (also used to indicate density ratio $\rho/\rho_R$ ).
$\theta$	Angle of attitude [ $\alpha = \sin \theta$ ; $\phi(\alpha) = \cos \theta$ ]
$\lambda$	Constant Lagrange multiplier
$\Lambda$	Functional to be minimized
$\mu$	Variable Lagrange multiplier
$v$	Independent variable for parametric formulation
$\Pi$	Euler-Lagrange sum in non-parametric form
$\rho$	Density $\left( \frac{\text{lb. sec.}^2}{\text{ft}^4} \right)$
$\tau = \frac{tg}{V_R}$	Non-dimensional time or independent variable
$\phi$	Constraints
$\varphi$	Characteristic line

$\Phi$	Equations of variation of the constraints
$\psi$	End-conditions
$\Psi$	Equations of variation of the end-conditions

### Superscripts

$$(\dots)' = \frac{d}{dt} (\dots)$$

$$(\dot{\dots}) = \frac{d}{dt} (\dots)$$

$$(\dots)^* = \text{Optimum value or index-value}$$

### Subscripts

I	Initial point
F	Final point
R	Reference value

## A. FIRST NECESSARY CONDITIONS FOR AN EXTREMAL

In this section some considerations on the first necessary conditions for an extremal will be made.

The proposed problem, of the Mayer form, is stated as that of "finding in the class of arcs

$$q_j(\tau), j = 1, \dots, m, \tau_I \leq \tau \leq \tau_F \quad (1)$$

satisfying a set of differential constraints

$$\phi_i(q'_j, q_j, \tau) = \phi_i(q'_1, \dots, q'_m, q_1, \dots, q_m, \tau) = 0, i = 1; \dots; n < m \quad (2)$$

one minimizing a function of the terminal conditions

$$\Lambda(q_{j_I}, q_{j_F}, \tau_I, \tau_F) = \Lambda[q_j(\tau_I), q_j(\tau_F), \tau_I, \tau_F] \quad (3)$$

while the end-constraints

$$\psi_\rho(q_{j_I}, q_{j_F}, \tau_I, \tau_F), \rho = 1, \dots, r \leq 2m + 1 \quad (4)$$

are made to vanish."

Some remarks are here in order. The constraints (2), may include some holonomic conditions, viz., for some  $i$ 's

$$\frac{\partial \phi_i}{\partial q'_1} = \dots = \frac{\partial \phi_i}{\partial q'_m} = 0.$$

Each of the non-holonomic constraints appearing in (2) does not necessarily include all the  $q'_j$ . The following derivations hold in any case. The end-constraints (4) for  $r < 2m + 1$  give what is known as a "variable end-point problem". For  $r = 2m + 1$ , a fixed end-point problem is formulated. If  $n < m$  the problem is said to have  $m - n$  "degrees of freedom", or degrees of indetermination. For the particular case  $m = n$  (zero degree of freedom) the problem is said to be determined. The variational problem may

degenerate then into an ordinary extremum associated with the "optimization of the boundary conditions". The latter occurrence depends on the boundary conditions (4) imposed and on the form of Eq. (2).

The summation convention of tensorial analysis will be freely used throughout. Also, for simplicity of nomenclature, whenever possible it will be understood that  $q = (q_1, \dots, q_m)$ . Subindexes will be used to indicate particular points or partial differentiation.

To derive the first necessary conditions for an extremal of the problem formulated by Eqs. (1) to (4) the family of admissible arcs is imbedded in a one-parameter family  $q_j(r, \epsilon)$ ,  $\tau_1(\epsilon) \leq r \leq \tau_F(\epsilon)$ , containing the normal extremal  $E$  for  $\epsilon = 0$ . It is assumed that  $E$  is an admissible arc with a set of admissible variations

$$\eta_j(r) = \left. \frac{\partial q_j(r, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0}, \quad \xi_1 = \left. \frac{d\tau_1(\epsilon)}{d\epsilon} \right|_{\epsilon=0}, \quad \xi_F = \left. \frac{d\tau_F(\epsilon)}{d\epsilon} \right|_{\epsilon=0} \quad (5)$$

with which it satisfies the equations of variation  $\Phi_i(\eta, \eta', r) = 0$  and  $\Psi_\rho(\eta_1, \eta_F, \xi_1, \xi_F) = 0$ . For other hypotheses on the arc whose minimizing properties are considered [e.g., normality relative to the end conditions, continuity of the state functions  $q_j$  and piecewise continuity of  $q'_j$  between corners, existence and properties of the partial derivatives of  $\phi_i$  in the neighborhood of the elements  $(q_j, q'_j, r) \in E$ , existence and properties of the partial derivatives of  $\Lambda$  and  $\psi_\rho$  in the neighborhood of the end values  $(q_{j_F}, q_{j_1}, \tau_1, \tau_F) \in E$ ], the reader is referred to the available literature on the theory of the Calculus of Variations (Refs. 1 to 6).

The developments to follow are aimed to obtain the first necessary conditions for an extremal in a simple manner, preserving the necessary rigorousness of formulation while providing an organized analysis and fast reference for the reader.

The first necessary conditions for a minimum of the function  $\Lambda$  may be derived by writing the minimal problem in the formally non-parametric Bolza form

$$J = \lambda_0 \Lambda + \lambda_\rho \psi_\rho + \int_{\tau_1}^{\tau_F} \Pi(q'_j, q_j, \mu_i, r) dr = \min \quad (6)$$

where  $\Pi$  is the Euler-Lagrange sum  $\mu_1(\tau)\phi_1 + \dots + \mu_n(\tau)\phi_n$ ,  $\mu_i(\tau)$  is a set of variable Lagrange multipliers and  $\lambda_0, \lambda_\rho, \rho = 1, \dots, r \leq 2m+1$ , is a set of constant multipliers. For the one-parameter admissible family  $q_j(\epsilon, \tau), \tau_I(\epsilon) \leq \tau \leq \tau_F(\epsilon)$ . Eq. (6) gives

$$\begin{aligned}
 J(\epsilon) = & \lambda_0 \Lambda \left\{ q_j \left[ \tau_I(\epsilon), \epsilon \right], q_j \left[ \tau_F(\epsilon), \epsilon \right], \tau_I(\epsilon), \tau_F(\epsilon) \right\} \\
 & + \lambda_\rho \psi_\rho \left\{ q_j \left[ \tau_I(\epsilon), \epsilon \right], q_j \left[ \tau_F(\epsilon), \epsilon \right], \tau_I(\epsilon), \tau_F(\epsilon) \right\} \\
 & + \int_{\tau_I(\epsilon)}^{\tau_F(\epsilon)} \Pi \left[ q'_j(\tau, \epsilon), q_j(\tau, \epsilon), \tau \right] d\tau
 \end{aligned} \tag{7}$$

where  $|\epsilon|$  is assumed sufficiently small so that all the members of the admissible family  $q_j(\tau, \epsilon)$  lie in an arbitrarily small neighborhood of the extremal  $q_j(\tau)$ . Thus,  $J(\epsilon)$  must have a minimum for  $\epsilon=0$  relative to all values of  $\epsilon$  in the neighborhood of  $\epsilon=0$ . Then, the necessary condition for a minimum is

$$\left. \frac{dJ(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0 \tag{8}$$

The minimal requirement in Eq. (6) is meaningful even if some of the  $q_j$  functions [assuming they do not appear differentiated in Eq. (2)] experience finite jumps. Admissible functions  $q_j$  of this type lead to problems called with "discontinuous control". The solution in the  $q(\tau)$  state variables is then called a "broken solution" and the extremal is composed of sub-arcs between the terminal points.

From Eqs. (7) and (8) it is then obtained

$$\begin{aligned}
 dJ \Big|_{\epsilon=0} = \frac{dJ(\epsilon)}{d\epsilon} \Big|_{\epsilon=0} d\epsilon = \lambda_0 \left( \Lambda_{q_{I,F}} dq_{I,F} + \Lambda_{r_{I,F}} dr_{I,F} \right) \\
 + \lambda_\rho \left( \psi_{q_{I,F}} dq_{I,F} + \psi_{r_{I,F}} dr_{I,F} \right) + \\
 + \left[ \Pi_q, dq + (\Pi - q' \Pi_q), dr \right]_I^F - \int_{r_I}^{r_F} [\Pi]_q \delta q dr = 0
 \end{aligned} \quad (9)$$

where for simplicity of notation the following replacements have been introduced

$$\Lambda_{q_{I,F}} dq_{I,F} = \Lambda_{q_{j_I}} dq_{j_I} + \Lambda_{q_{j_F}} dq_{j_F}, \quad \Lambda_q = \frac{\partial \Lambda}{\partial q}$$

$$\Lambda_{r_{I,F}} dr_{I,F} = \Lambda_{r_I} dr_I + \Lambda_{r_F} dr_F, \quad \Lambda_r = \frac{\partial \Lambda}{\partial r}$$

$$\Pi_q, dq = \Pi_{q_j}, dq_j = \Pi_{q_1}, dq_1 + \dots + \Pi_{q_m}, dq_m, \quad \Pi_{q'} = \frac{\partial \Pi}{\partial q'}$$

$$\psi_{q_a} = - \frac{\partial \psi_\rho}{\partial q_a}, \quad a = I \text{ or } F$$

$$\left[ \dots \right]_I^F = \left[ \dots \right]_F - \left[ \dots \right]_I$$

$$\left[ \dots \right]_q = \frac{d}{dr} \left( \frac{\partial [\dots]}{\partial q'} \right) - \frac{\partial [\dots]}{\partial q}$$

$$\delta q_j = \epsilon \eta_j, \quad \epsilon \rightarrow d\epsilon$$

$$\delta q_{j_a} = \left[ dq_j - q' dr \right]_a, \quad a = I \text{ or } F.$$

Due to the arbitrariness of the variations  $\delta q$  consistent with the equations of variation  $\Phi_i(\delta q, \delta q', r) = 0$  and of the variations  $dq_I, dq_F, dr_I, dr_F$ , after simple considerations it is seen that the first necessary conditions for an extremal, from Eq. (9), require

a) along the extremal arc E the variational derivatives  $[\Pi]_q$  must vanish, according to the fundamental lemma of the Calculus of Variations.

b) at the terminal points the  $[(r+1) \times (2m+2)]$  - matrix

$$\left\| \begin{array}{cccc} \lambda_0 \frac{\partial \Lambda}{\partial q_I} - B_{jI} & \lambda_0 \frac{\partial \Lambda}{\partial q_F} B_{jF} & \lambda_0 \frac{\partial \Lambda}{\partial r_I} - A_I & \lambda_0 \frac{\partial \Lambda}{\partial r_F} + A_F \\ \frac{\partial \psi_\rho}{\partial q_I} & \frac{\partial \psi_\rho}{\partial q_F} & \frac{\partial \psi_\rho}{\partial r_I} & \frac{\partial \psi_\rho}{\partial r_F} \end{array} \right\| \quad (10)$$

must have rank  $R < r+1$ . In the matrix (10),

$$B_{ja} = \Pi_{q_j}, \quad \left| \begin{array}{c} \\ \\ \end{array} \right|_a \quad \text{and} \quad A_a = \left[ \Pi - q_j' \Pi_{q_j'} \right]_a, \quad a = I \text{ or } F.$$

Eq. (10) is the matrix form of the "Transversality Condition".

From Eq. (9) it can be seen that the Transversality Condition in its expanded form may be also written

$$\lambda_0 d\Lambda + \left[ \Pi_{q_j} dq_j + \left( \Pi - q_j' \Pi_{q_j'} \right) dr \right]_I^F = 0 \quad (11)$$

for any admissible set of differentials  $dq_I, dq_F, dr_I, dr_F$  satisfying the equations of variation on E.

$$\Psi_\rho(dq_I, dq_F, dr_I, dr_F) = 0.$$

Therefore, for a minimizing arc E there must exist  $n$  functions  $\mu_i(r)$  not all identically zero in the interval  $r_I \leq r \leq r_F$  satisfying the

condition that the variational derivatives  $[\Pi]_q$  vanish along  $E$  and making all determinants of order  $r+1$  of the matrix (10) vanish. The preceding statement is equivalent to saying that there must exist  $n$  functions  $\mu_i(\tau)$  not all identically zero in the interval  $\tau_1 \leq \tau \leq \tau_F$  satisfying the equations  $[\Pi]_q = 0$  on  $E$ , and  $r+1$  constants  $\lambda_0, \lambda_\rho$  not all zero satisfying the equations

$$\lambda_0 \Lambda_{q_1} - B_{j_1} + \lambda_\rho \psi_{q_1} = 0$$

$$\lambda_0 \Lambda_{q_F} + B_{j_F} + \lambda_\rho \psi_{q_F} = 0$$

$$\lambda_0 (\Lambda_{q_1} q'_1 + \Lambda_{\tau_1}) - \Pi_1 + \lambda_\rho (\psi_{q_1} q'_1 + \psi_{\tau_1}) = 0$$

$$\lambda_0 (\Lambda_{q_F} q'_F + \Lambda_{\tau_F}) + \Pi_F + \lambda_\rho (\psi_{q_F} q'_F + \psi_{\tau_F}) = 0 \quad (12)$$

An admissible arc [viz, an arc satisfying Eqs. (2) and (4)] in the interval  $\tau_1 \leq \tau \leq \tau_F$  with a set of constants  $\lambda_0, \lambda_1, \dots, \lambda_r \neq (0, 0, \dots, 0)$  and a sum  $\Pi = \mu_i(\tau) \phi_i$  with multipliers  $[\mu_1(\tau), \dots, \mu_n(\tau)] \neq [0, \dots, 0]$  and satisfying the Euler equations and Eq. (12) with these constants and multipliers is said to satisfy the Multiplier Rule.

If the one-parameter family  $q(\tau, \epsilon)$ , for  $\epsilon=0$ , contains the extremal arc having a corner at  $\tau=\tau_c$ , then other necessary conditions must be added to the previous ones. However, if the same one-parameter family is applied to derive necessary conditions at a corner it will be seen that the necessary vertex conditions may be only partially obtained due to the discontinuity in the slope  $q'_j$  at such point. In other words, along the so-called "line of corners" of the family  $q_j(\tau, \epsilon)$  defined by  $q_c = q[\tau_c(\epsilon), \epsilon]$  in general,  $dq_c(\tau_c - 0) \neq dq_c(\tau_c + 0)$ . Therefore, it is convenient to write the minimal requirement of Eq. (6) in a formally parametric Bolza form. To this extent a parametric representation  $\tau = \tau(\nu)$  is introduced. The derivative  $\dot{\tau} = d\tau / d\nu$  is taken  $\dot{\tau} > 0$  for any  $\nu_1 \leq \nu \leq \nu_F$  not to alter the sense of traversal on the extremal.

The admissible family taken now is  $q_j = q_j(\nu, \epsilon)$  and it contains the extremal for  $\epsilon = 0$ . The functions  $q_j$  are continuous in the interval  $\nu_I \leq \nu \leq \nu_F$  while the functions  $\dot{q}_j$  may be piecewise continuous. Assume first that there are no corners in  $\nu_I \leq \nu \leq \nu_F$ . Eq. (18) is now written in parametric form as follows

$$J = \lambda_0 \Lambda + \lambda_\rho \psi_\rho + \int_{\nu_I}^{\nu_F} F(\mu_i, \dot{q}_j, q_j, \dot{r}, r) d\nu \quad (13)$$

The functions  $\phi_i [\dot{q}_j, q_j, \dot{r}, r]$  have continuous partial derivatives of the first three orders in a neighborhood of the values  $(\dot{q}_j, q_j, \dot{r}, r)$  on  $E$ . The functions  $\psi_\rho [q_j(\nu_I), q_j(\nu_F), r(\nu_I), r(\nu_F)]$  have continuous partial derivatives of the first two orders in a neighborhood of the end values  $(q_{j_I}, q_{j_F}, r_I, r_F)$ . In Eq. (13) the integrand  $F = \Pi \dot{r}$  is homogeneous of order one in the derivatives  $\dot{q}_j, \dot{r}$ . Then, the variational problem leads to an extremal solution independent of the choice of parameter since  $t > 0$  and  $F$  is homogeneous (Ref. 2).

For the parametric admissible family  $q_j = q_j(\nu, \epsilon)$ ,  $r = r(\nu, \epsilon)$  adopted, the variations are

$$\delta q_j = \left. \frac{\partial q_j}{\partial \epsilon} \right|_{\epsilon=0} \epsilon, \quad \delta \dot{q}_j = \left. \frac{\partial \dot{q}_j}{\partial \epsilon} \right|_{\epsilon=0} \epsilon, \quad \delta r = \left. \frac{\partial r}{\partial \epsilon} \right|_{\epsilon=0} \epsilon,$$

where  $\epsilon \rightarrow d\epsilon$ . The absolute value  $|\epsilon|$  is assumed sufficiently small so all the members of the admissible family  $q_j(\nu, \epsilon)$  lie in an arbitrarily small neighborhood of the extremal  $q_j(\nu)$ . It is now assumed that at the terminal points

$$\left. \frac{d\nu}{d\epsilon} \right|_I = \left. \frac{d\nu}{d\epsilon} \right|_F = 0.$$

From Eq. (8) is now obtained

$$\begin{aligned} J(\nu_I, \nu_F, \epsilon) = & \lambda_0 \Lambda [q_j(\nu_I, \epsilon), q_j(\nu_F, \epsilon), r(\nu_I, \epsilon), r(\nu_F, \epsilon)] \\ & + \lambda_\rho \psi_\rho [q_j(\nu_I, \epsilon), q_j(\nu_F, \epsilon), r(\nu_I, \epsilon), r(\nu_F, \epsilon)] + \\ & + \int_{\nu_I}^{\nu_F} F[\dot{q}_j(\nu, \epsilon), q_j(\nu, \epsilon), \dot{r}(\nu, \epsilon), r(\nu, \epsilon)] d\nu \end{aligned} \quad (14)$$

Then, by hypothesis, for given  $\nu_1$  and  $\nu_F$ ,  $J(\nu_1, \nu_F, \epsilon)$  must have a minimum for  $\epsilon=0$  relative to all values of  $\epsilon$  in the neighborhood of  $\epsilon=0$ . The necessary condition for a minimum is consequently

$$\left. \frac{\partial J(\nu_1, \nu_F, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0} = 0 \quad (15)$$

After performing operations, from Eqs. (14) and (15) is found that the differential of  $J$  on the external ( $\epsilon=0$ ) leads to

$$\begin{aligned} & \lambda_0 \left[ \Lambda_{q_{1,F}} \delta q_{1,F} + \Lambda_{r_{1,F}} \delta r_{1,F} \right] + \lambda \rho \left[ \psi_{q_{1,F}} \delta q_{1,F} + \psi_{r_{1,F}} \delta r_{1,F} \right] \\ & + \left[ F_{\dot{q}} \delta q + F_{\dot{r}} \delta r \right]_1^F - \int_{\nu_1}^{\nu_F} [F]_q \delta q d\nu - \int_{\nu_1}^{\nu_F} [F]_r \delta r d\nu = 0 \end{aligned} \quad (16)$$

Now the symbol  $[\dots]_a$  indicates the variational derivative

$$\frac{d}{d\nu} (\dots)_a - \frac{\partial}{\partial a} (\dots)$$

The rest of the notation in Eq. (16) has been explained before.

Note now that  $F$  is homogeneous of degree one in the derivatives  $\dot{q}_j$  and  $\dot{r}$ , thus

$$\dot{q}_j \frac{\partial F}{\partial \dot{q}_j} + \dot{r} \frac{\partial F}{\partial \dot{r}} - F = 0 \quad (17)$$

Moreover, it may be found that

$$\begin{aligned} F_{q_j} &= \Pi_{q_j} \dot{r}, \quad F_{\dot{r}} = \Pi - q_j' \Pi_{q_j}, \quad F_{\dot{q}_j} = \Pi_{q_j} \\ [F]_{q_j} &= [\Pi]_{q_j} \dot{r}, \quad [F]_r = \left( \frac{d}{d\nu} (\Pi - q_j' \Pi_{q_j}) - \Pi_r \right) \dot{r} \end{aligned}$$

thus, replacing in Eq. (16) the following is obtained

$$\begin{aligned}
 \delta J = & \lambda_0 \left[ \Lambda_{q_{I,F}} dq_{I,F} + \Lambda_{r_{I,F}} dr_{I,F} \right] \\
 & + \lambda \rho \left[ \psi_{q_{I,F}} dq_{I,F} + \psi_{r_{I,F}} dr_{I,F} \right] \\
 & + \left[ \Pi_{q_j} dq_j + \left( \Pi - q'_j \Pi_{q'_j} \right) dr \right]_I^F - \int_{\nu_I}^{\nu_F} \left[ \Pi \right]_{q_j} \dot{r} \delta q_j d\nu \\
 & - \int_{\nu_I}^{\nu_F} \left( \frac{d}{dr} \left( \Pi - q'_j \Pi_{q'_j} \right) - \Pi_r \right) \dot{r} \delta r d\nu = 0
 \end{aligned} \tag{18}$$

where, since  $\frac{d\nu_I}{d\epsilon} = \frac{d\nu_F}{d\epsilon} = 0$ , then  $dq_a = \delta q_a$  and  $dr_a = \delta r_a$ ,  $a = I$  or  $F$ .

Due to the arbitrariness of the variations  $\delta q_j, \delta r$  (satisfying the equations of variation  $\Phi_i = 0$ ) and the arbitrariness of the differentials  $dq_{I,F}, dr_{I,F}$  the same necessary conditions for an extremal are immediately obtained as before, i.e., as for the non-parametric problem previously considered. These conditions are given by the Euler equations  $[\Pi]_q = 0$  along  $E$ , and the Transversality Condition in the matrix (10). However, one more important equation is obtained from the last integral on the right-hand side of Eq. (18) as a benefit of the parametric representation applied. All previous considerations on the necessary conditions for an extremal hold here consequently, and in addition it is required that along  $E$  for  $\tau_I \leq \tau \leq \tau_F$ , the equation

$$\frac{d}{dr} (\Pi - q'_j \Pi_{q'_j}) - \Pi_r = 0 \tag{19}$$

must be satisfied.

Assume now that there exists a corner on  $E$ . That is, the  $q_j(\nu)$  are continuous on  $E$  while some or all of the  $q_j(\nu)$  are piecewise continuous on  $E$ . Since  $\dot{r} > 0$  and continuous on  $E$  the previous considerations similarly apply to the corresponding  $q_j(\tau)$  and  $q'_j(\tau)$ . The line of corners on the admissible family, satisfies the condition  $d\nu_c / d\epsilon = 0$ . The admissible families assumed, for the parametric and non-parametric cases are represented in Figs. (1a) and (1b).

By means of simple replacements it may be found from Eqs. (14) and (15) that, for the case of broken extremals, Eq. (16) has a similar form but where the last two terms in the first member are replaced by (here is assumed only one corner, the extension follows immediately)

$$\begin{aligned} \left[ F_{\dot{q}} \delta q + F_{\dot{r}} \delta r \right]_{c+0}^{c-0} &= \int_{\nu_I}^{\nu_c-0} [F]_q \delta q d\nu - \int_{\nu_I}^{\nu_c-0} [F]_r \delta r d\nu \\ &- \int_{\nu_c+0}^{\nu_F} [F]_q \delta q d\nu - \int_{\nu_c+0}^{\nu_F} [F]_r \delta r d\nu \end{aligned} \quad (20)$$

In Eq. (20),  $c-0$  and  $c+0$  indicate limiting values for  $\nu$  approaching  $\nu_c$  from  $\nu < \nu_c$  and  $\nu > \nu_c$  respectively. Eq. (16) with the last two terms in the first member replaced by those given in Eq. (20) leads, after the usual considerations on the variations, to the following conditions along E and at c

$$[F]_{q_j} = 0, \text{ for } \nu_I \leq \nu \leq \nu_c - 0 \text{ and } \nu_c + 0 \leq \nu \leq \nu_F \quad (21)$$

$$[F]_r = 0, \text{ for } \nu_I \leq \nu \leq \nu_c - 0 \text{ and } \nu_c + 0 \leq \nu \leq \nu_F \quad (22)$$

$$\left( F_{\dot{q}} \right)_{c+0}^{c-0} dq_c = 0, \quad \left( F_{\dot{r}} \right)_{c+0}^{c-0} dr_c = 0 \quad (23)$$

At the end points I and F the same Transversality Condition as before is also obtained. To derive Eq. (23) account has been taken of the fact that

$$\frac{d\nu_c}{d\epsilon} = 0, \quad dq_c = \delta q_c, \quad dr_c = \delta r_c$$

and the variations are continuous at c. While some of the necessary conditions may be derived using the non-parametric formulation, the parametric form used here offers the advantage of giving, in addition to Eq. (22), viz.,

$$[F]_r = \left( \frac{d}{dr} (\Pi - \Pi_{q_j} q'_j) - \Pi_r \right) \dot{r} = 0, \quad (\dot{r} > 0)$$

the complete set of the so-called Erdmann-Weierstrass vertex conditions, given by

$$\Pi_{q_j'} \Big|_{c-o} - \Pi_{q_j'} \Big|_{c+o} = 0, \quad j = 1; 2; \dots; m \quad (24)$$

$$\left( \Pi - q_j' \Pi_{q_j'} \right) \Big|_{c-o} - \left( \Pi - q_j' \Pi_{q_j'} \right) \Big|_{c+o} = 0, \quad j = 1, \dots, m \quad (25)$$

in view of the arbitrariness of  $dq_c$  and  $d\tau_c$  in Eq. (23).

Summarizing, from previous developments it is concluded that the first necessary conditions for an extremal in the class of arcs  $q_j(\tau)$ ,  $\tau_1 \leq \tau \leq \tau_F$  satisfying a set of equations of the form (2), (4) are that the Euler-Lagrange sum  $\Pi(q_j', q_j, \mu_i, \tau) = \mu_i(\tau) \phi_i$  satisfies the equations

$$[\Pi]_{q_j} = 0, \quad \Pi - \Pi_{q_j'} q_j' = \int_{\tau_1}^{\tau} \Pi_{\tau} d\tau + \text{const.} \quad (26)$$

along every sub-arc composing the external arc, with a set of  $n$  non-simultaneously vanishing multipliers  $\mu_i(\tau)$  on  $\tau_1 \tau_F$  such that the associated matrix (10) is of rank  $R < r+1$ , and satisfying at junctions of different sub-arcs the vertex conditions (24) and (25).

#### 1. Non-Singularity and Normality.

The first necessary conditions previously derived apply to an extremal arc  $E$  composed of sub-arcs  $E^1, E^2, \dots, E^n$  on each of which  $q_j'$  is continuous and  $q_j'(\tau)$  and  $\mu_i(\tau)$  have at least first order derivatives with respect to  $\tau$ . Such an arc is then said to be non-singular, viz., along each sub-arc the functional determinant

$$\begin{vmatrix} \Pi_{q_j' q_{\ell}'} & \phi_{i q_j'} \\ \phi_{\beta q_{\ell}'} & 0 \end{vmatrix} \quad \begin{matrix} i, \beta = 1, \dots, n \\ j, \ell = 1, \dots, m \end{matrix} \quad (27)$$

is different from zero and at corners the Erdmann-Weierstrass conditions are satisfied. Condition (27) is an important necessary requirement in the study of sufficiency conditions. Furthermore, it is also necessary that there is no set of constants  $\lambda_0 = 0, \lambda_{\rho}$ ,  $\rho = 1, \dots, r \leq 2m+1$  with which Eqs. (12) are satisfied. Thus, the arc  $E$  is required to be non-singular and moreover normal, viz., the rank of the matrix

$$\left\| \begin{array}{cccc} -B_{j_I} & B_{j_F} & -\Pi_I & \Pi_F \\ \psi_{q_I} & \psi_{q_F} & \psi_{q_I} q'_I + \psi_{r_I} & \psi_{q_F} q'_F + \psi_{r_F} \end{array} \right\| \quad (28)$$

must be  $R=r+1$ . For a normal arc the sets of constant multipliers  $\lambda_o, \lambda_\rho$  and variable multipliers  $\mu_i(r)$  are unique. Therefore, accounting for the homogeneity of Eq. (26) in the  $\mu_i$ 's and from Eq. (12) it is seen that the unique set of constant multipliers may be always reduced, for a normal non-singular extremal, to one of the form  $(\lambda_o, \lambda_\rho) = (1, \lambda_\rho^*)$  where  $\lambda_\rho^*$  is the ratio  $\lambda_\rho / \lambda_o$  with a corresponding unique set of functions  $\mu_i^*(r)$  where  $\mu_i^* = \mu_i / \lambda_o$ .

## 2. The Weierstrass, Legendre and Clebsch-Mayer Necessary Conditions.

As indicated by Bliss in Ref. 1, Weierstrass noticed that not all variations have the property that their derivative tends to zero with the variations themselves when the parameter approaches zero. The type of variations approaching zero with their derivatives when  $\epsilon \rightarrow 0$  has been assumed in deriving the first necessary conditions in the preceding paragraphs. The type of Weierstrassian variations will be applied now in deriving the local Weierstrass necessary condition. To give a simple example of the latter type of variations consider  $\eta$ -variations of the form

$$\begin{aligned} \eta &\equiv 0, \quad r \leq r^* - 0 \\ \eta(\nu, r) &= -\Delta q' \left( 1 - \frac{r - r^*}{\nu} \right), \quad r^* + 0 \leq r \leq r^* + \nu \\ \eta &\equiv 0, \quad r \geq r^* + \nu \end{aligned}$$

Then, in the interval  $r^* \leq r \leq r^* + \nu$  it is  $\eta(r^*) = -\Delta q'$ , where  $\Delta q'$  indicates a difference of slopes  $q' - q'^*$  at  $r = r^*$ . Also, in the same interval  $\eta'(r^*) = \Delta q' / \nu$ . The associated varied functions may be written in terms of a two-parameter family

$$\begin{aligned} q(\epsilon, \nu, r) &= q^*(r) + \epsilon \eta(\nu, r) \\ q'(\epsilon, \nu, r) &= q'^*(r) + \epsilon \eta'(\nu, r) \end{aligned}$$

where the  $\delta$ -variations are  $\delta q = \epsilon \eta$ ,  $\delta q' = \epsilon \eta'$ ,  $\epsilon \rightarrow d\epsilon$ . Note that when  $\epsilon$  tends to zero with  $\nu$  [e.g.,  $\epsilon = \epsilon(\nu) = \nu$ ,] the variations give at  $\tau = \tau^*$  the following values

$$\lim_{\nu \rightarrow 0} \epsilon \eta = 0, \quad \lim_{\nu \rightarrow 0} \epsilon \eta' = \Delta q'$$

Thus the  $\delta q'$  variations do not approach zero with the variations  $\delta q$  when the parameter  $\epsilon$  approaches zero.<sup>(\*)</sup> Similar variations, of the Weierstrassian type, will be implied in the following developments. As schematically shown in Fig. 2 the arcs considered are assumed to be composed of sub-arcs of three different adjacent families,

- a)  $q(\tau, \epsilon)$ ,  $\tau_1(\epsilon) \leq \tau \leq \tau_c$ , one-parameter admissible family in the neighborhood of the extremal E, ( $\epsilon$  arbitrarily small), containing the piece IC of the normal extremal E(IF) for  $\epsilon = 0$ .
- b)  $Q(\tau, \epsilon)$ ,  $\tau_c \leq \tau \leq \tau_c + \nu$ , one-parameter family near C containing the differentiable admissible\* arc C for  $\epsilon = 0$ , (and point C on E for  $\epsilon = 0$ ,  $\nu = 0$ ).
- c)  $q(\tau, \epsilon, \nu)$ ,  $\tau_c + \nu \leq \tau \leq \tau_F(\epsilon)$ , two-parameter admissible family of arcs E' joining points of the arc C (points of C in a neighborhood of E, viz.,  $\nu$  is arbitrarily small) with the end line  $\tau_F(\epsilon)$ , such that for  $\epsilon = 0$ ,  $\nu = 0$ , contains the piece CF of the normal extremal E(IF).

When the above functions are substituted in the sum J, [given in Eq. (6)] then the two-parameter sum may be written

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(\*) Other interesting examples of Weierstrassian, or "strong" variations, are given by Bolza in Ref. 4.

$$\begin{aligned}
J(\epsilon, \nu) = & \lambda_0 \Lambda \left\{ q[r_F(\epsilon), \epsilon, \nu], q[r_I(\epsilon), \epsilon], r_I(\epsilon), r_F(\epsilon) \right\} + \\
& + \lambda \rho \psi \rho \left\{ q[r_F(\epsilon), \epsilon, \nu], q[r_I(\epsilon), \epsilon], r_I(\epsilon), r_F(\epsilon) \right\} + \\
& + \int_{r_I(\epsilon)}^{r_c} \Pi[r, q(r, \epsilon), q'(r, \epsilon)] dr + \int_{r_c}^{r_c + \nu} \Pi[r, Q, Q'] dr + \\
& + \int_{r_c + \nu}^{r_F(\epsilon)} \Pi[r, q(r, \epsilon, \nu), q'(r, \epsilon, \nu)] dr
\end{aligned} \tag{29}$$

The normal extremal arc  $E$  has therefore been imbedded in a special family of arcs containing  $E$  ( $E'$ ) for  $\epsilon = 0$ ,  $\nu = 0$ . Assuming a dependence  $\epsilon = \epsilon(\nu)$  such that  $\epsilon \rightarrow 0$  for  $\nu \rightarrow 0$ , the admissible family of adjacent arcs, defined in the adjoining intervals  $(r_I, r_c)$ ;  $(r_c, r_c + \nu)$ ;  $(r_c + \nu, r_F)$  will give for the sum  $J(\epsilon, \nu) = J[\epsilon(\nu), \nu]$  the derivative

$$\hat{J}(0) = \frac{d}{d\nu} J \bigg|_{\nu=0} = J_\epsilon(0, 0) \hat{\epsilon}(0) + J_\nu(0, 0) \tag{30}$$

where  $\hat{\epsilon}(0) = \frac{d\epsilon}{d\nu} \bigg|_{\nu=0} > 0$ . Since  $J$ , by hypothesis, affords a relative

minimum for  $\epsilon = 0, \nu = 0$ , then the value of the sum  $J$  for any  $\nu$  in the neighborhood of  $\nu = 0$  must not decrease when  $\nu$  increases from  $\nu = 0$ . Note that  $\nu$  is assumed arbitrarily small but must be  $\nu \geq 0$ , since for values  $\nu < 0$  the arcs would have three points on each coordinate of the interval  $r_c + \nu < r < r_c$ . Then, from previous considerations is deduced that it must be  $\frac{d}{d\nu} J \bigg|_{\nu=0} \geq 0$ .

From Eq. (29) performing operations and accounting for the conditions obtained in previous paragraphs it may be found that the requirement  $\hat{J}(0) \geq 0$  leads to

$$\Pi(\mu, \tau, Q, Q') \Big|_{\tau_c} - \Pi(\mu, \tau, q, q') \Big|_{\tau_c} - \Pi_{q'}(\mu, \tau, q, q') \bar{\eta}(\tau) \Big|_{\tau_c} \geq 0 \quad (31)$$

where  $\bar{\eta}(\tau) = \frac{\partial q(\tau, \epsilon, \nu)}{\partial \nu} \Big|_{\nu=0} = \bar{\eta}(\tau, 0, 0)$

Now, at any point D of the admissible family (Fig. 2) it is

$$Q(\tau_c + \nu, \epsilon) - q(\tau_c + \nu, \epsilon, \nu) = 0 \quad (32)$$

[on E,  $\nu = 0$ ,  $Q(\tau_c, 0) = q(\tau_c, 0, 0)$ ] and then the derivatives at  $\nu = 0$  ( $\tau = \tau_c$ ), i.e. on E satisfy

$$Q'(\tau_c) + Q_\epsilon(\tau_c, 0) \hat{\epsilon}(0) - q'(\tau_c) - q_\epsilon(\tau_c, 0, 0) \hat{\epsilon}(0) - \bar{\eta}(\tau_c) = 0 \quad (33)$$

Also, at any point C of the admissible family,  $Q(\tau_c, \epsilon) - q(\tau_c, \epsilon) = 0$ , and therefore on E ( $\nu = 0$ )

$$Q_\epsilon(\tau_c, 0) - q_\epsilon(\tau_c, 0) = 0 \quad (34)$$

Then, from Eqs. (31) to (34) is obtained

$$\Pi(\mu, \tau, q, Q') - \Pi(\mu, \tau, q, q') - (Q' - q') \Pi_{q'}(\mu, \tau, q, q') \geq 0 \quad (35)$$

at the time  $\tau = \tau_c$  on the normal extremal E.

Eq. (35) is the local Weierstrass necessary condition to be satisfied by any normal extremal arc E in addition to the first necessary conditions analyzed previously. Expanding the first two terms in the left hand side of Eq. (35) in Taylor series in a close neighborhood of the values  $(\mu, \tau, q, q') \in E$  and for increments  $Q' - q' = \delta q'$  such that powers higher than the first are negligible, it is easily found

$$\Pi_{q_j, q_k}(\mu, \tau, q, q') (Q'_j - q'_j) (Q'_k - q'_k) \geq 0, \quad j, k = 1, \dots, m \quad (36)$$

which is the necessary condition to be satisfied at any given  $\tau$  on  $E$  as originally derived by Legendre. The condition in Eq. (36) as generalized by Clebsch and Mayer states that at each element  $(\mu, \tau, q, q')$  of a normal external arc the inequality

$$\Pi_{q'_j q'_k}(\mu, \tau, q, q') \delta q'_j \delta q'_k \geq 0 \quad (37)$$

must be satisfied by every set  $(\delta q'_j) \neq (0)$  consistent with the equations of variation

$$\Phi_i \equiv \phi_{i q'_j} \delta q'_j = 0, \quad \begin{matrix} i = 1; 2; \dots; n \\ j = 1, \dots, m \end{matrix} \quad (38)$$

The Clebsch condition may be written in its generalized form by considering that some control variables  $q_\ell$  (non-differentiated  $q$ 's) may be present in  $\Pi$ . In such case, the Clebsch condition requires that at each element  $(\mu, \tau, q_j, q'_j, q_\ell)$  of a normal extremal arc the inequality

$$\begin{aligned} \Pi_{q_\ell q_p}(\mu, \tau, q, q') \delta q_\ell \delta q_p + \Pi_{q_\ell q'_j}(\mu, \tau, q, q') \delta q_\ell \delta q'_j + \\ + \Pi_{q'_j q'_k}(\mu, \tau, q, q') \delta q'_j \delta q'_k \geq 0, \quad \begin{matrix} j, k = 1, \dots, m \\ \ell, p = m+1, \dots, s \end{matrix} \end{aligned} \quad (39)$$

must be satisfied by every set  $(\delta q'_j, \delta q_\ell) \neq (0, 0)$  consistent with the equations of variations

$$\Phi_i \equiv \phi_{i q_\ell} \delta q_\ell + \phi_{i q'_j} \delta q'_j = 0, \quad \begin{matrix} i = 1; 2; \dots; n \\ \ell = m+1, \dots, s \\ j = 1, \dots, m \end{matrix} \quad (40)$$

## B. LEGENDRE TRANSFORMATION OF THE VARIATIONAL PROBLEM INTO A GENERALIZED CANONICAL FORM

For the sake of generality of formulation, it will be here assumed that the Mayer problem formulated in Section 1 is expressed in the class of arcs

$$q_j(r), \quad j = 1, \dots, m, \quad r_1 \leq r \leq r_F$$

$$q_\ell(r), \quad \ell = m+1, \dots, s$$

satisfying a set of differential constraints

$$\phi_i(r, q_j, q'_j, q_\ell) = \phi_i(r, q_1, \dots, q_m, q'_1, \dots, q'_m, q_{m+1}, \dots, q_s) = 0, \quad \begin{matrix} i = 1; \dots; n \\ j = 1, \dots, m \leq n \\ \ell = m+1, \dots, s \end{matrix} \quad (41)$$

and terminal conditions as given by Eq. (4). The argument functions  $q_j(r)$  are called state variables and the functions  $q_\ell(r)$  are called control variables. It is simple to see that the first necessary conditions for an extremal derived in Section 1 and paragraphs 1.1 and 1.2 will remain formally the same. Evidently, the Euler equations associated with the variations of the control functions are to be added to the set derived in Section 1. These equations will be considered in the sequel.

In the developments to be considered in this Section, use will be made of the general variational principle that says (Ref. 2): "If a functional  $J$  is stationary for an admissible set of functions  $q_j(r)$  satisfying certain subsidiary conditions ( $\phi_i = 0, \psi_\rho = 0$ ), then  $J$  remains stationary for the arc  $q_j(r)$  when the subsidiary conditions are enlarged to include any further relation already satisfied by the functions  $q_j(r)$ ". The Lagrange multiplier method leads to the transformation considered in the following, thus allowing, in general, to express the extremal in terms of a set of first order differential equations instead of the second order Euler differential equations.

To this extent the so-called canonical variables  $(\bar{\tau}, q_j, q_\ell, v_j)$  related to the variables  $(\tau, q_j, q'_j, q_\ell, \mu_i)$  by the equations

$$v_j = \Pi_{q'_j}(\tau, q_j, q'_j, q_\ell, \mu_i), \quad 0 = \phi_i(\tau, q_j, q'_j, q_\ell) \quad (42)$$

are introduced, where  $\Pi = \mu_i \phi_i(\tau, q_j, q'_j, q_\ell)$  as defined before.

The extremal arc  $E$  is assumed to be composed of extremal sub-arcs  $E^1, E^2, \dots, E^n$ , to be normal [viz., satisfies  $\phi_i = 0$  and the Euler equations Eqs. (26) with a unique set of multipliers  $\lambda_0 = 1, \mu_i(\tau)$ ] to be non-singular in each sub-arc [viz.,  $q'_j(\tau)$  and  $\mu_i(\tau)$  have continuous derivatives between corners] along which  $q_\ell(\tau)$  and  $q'_\ell(\tau)$  are continuous, and satisfying at the terminal points the conditions  $\psi_\rho = 0$  and condition (11), (the functions  $q_j$  are continuous in  $\tau_1, \tau_F$  and the slopes  $q'_j$  are continuous on each sub-arc). Thus along  $E$ , accounting for the vertex conditions Eqs. (24) and the previous considerations, Eqs. (42) define functions  $v_j(\tau)$  continuous in  $(\tau_1, \tau_F)$  and having continuous derivatives  $v'_j(\tau)$  between corners. By implicit function theorems (Ref. 1) there exists a neighborhood  $N$  of  $(\tau, q_j, q_\ell, v_j) \in E$  in which Eqs. (42) have solutions

$$q'_j = \bar{q}'_j(\tau, q_j, q_\ell, v_j), \quad \mu_i = \bar{\mu}_i(\tau, q_j, q_\ell, v_j) \quad (43)$$

which reduce to  $q'_j(\tau), \mu_i(\tau)$  for  $[\tau, q_j(\tau), q_\ell(\tau), v_j(\tau)] \in E$ .

It is assumed that Eqs. (42) define a one-to-one correspondence between a region  $M$  of interior points  $(\tau, q_j, q'_j, q_\ell, \mu_i)$  and the region  $N$  of points  $(\tau, q_j, q_\ell, v_j)$ . Then, in view of previous considerations, Eqs. (42) have single-valued solutions of the form given by Eqs. (43), except possibly at corners. The generalized canonical transformation described is represented in Fig. 3.

According to the general principle the integrand function  $\Pi$  is now enlarged to form the following function

$$F = \Pi + \Pi_{q'_j} \left( \frac{dq_j}{d\tau} - \bar{q}'_j \right)$$

which, with obvious transformations, is written

$$F(r, q_j, q_\ell, v_j) \equiv v_j \frac{dq_j}{dr} - H \quad (44)$$

The Hamiltonian function  $H$  is (Refs. 1, 2, 8)

$$H(r, q_j, q_\ell, v_j) = v_j \bar{q}_j' - \Pi(r, q_j, q_\ell, \bar{q}_j', \bar{\mu}_i) \quad (45)$$

which is also written

$$H = \left[ q_j' \Pi_{q_j'} - \Pi(r, q_j, q_\ell, q_j', \mu_i) \right] \quad q_j' = \bar{q}_j'; \mu_i = \bar{\mu}_i \quad (46)$$

The derivatives of the Hamiltonian  $H$ -function are readily found to be

$$\begin{aligned} H_{q_j} &= -\Pi_{q_j}(r, q_j, q_\ell, \bar{q}_j', \bar{\mu}_i) ; \quad H_r = -\Pi_r(r, q_j, q_\ell, \bar{q}_j', \bar{\mu}_i) \\ H_{q_\ell} &= -\Pi_{q_\ell}(r, q_j, q_\ell, \bar{q}_j', \bar{\mu}_i) ; \quad H_{v_j} = \bar{q}_j'(r, q_j, q_\ell, v_j) \end{aligned} \quad (47)$$

Now, for the functional  $F(r, q_j, q_\ell, v_j)$  the corresponding equations of variation are (assuming unrestricted variations)

$$[F]_{q_j} \equiv v_j' + H_{q_j} = 0 \quad (48)$$

$$[F]_{q_\ell} \equiv H_{q_\ell} = 0 \quad (49)$$

$$[F]_{v_j} \equiv q_j' - H_{v_j} = 0 \quad (50)$$

where  $[...]_a$  indicates the variational derivative  $d/dr [(.....)_a] - (.....)_a$ . The extremal arc  $q_j(r), q_\ell(r), v_j(r)$ , must satisfy the end-conditions  $\psi_\rho = 0$  and the Transversality condition (canonical form)

$$d\Lambda(r_I, r_F, q_{j_I}, q_{j_F}) + \left[ v_j dq_j - H dr \right]_I^F = 0 \quad (51)$$

to be terminally admissible. Eq. (51) provides the so-called natural end-conditions for some or all of the  $v_j$ 's and for  $H$ .

The set of equations (48) to (50) is called canonical equations of the extremals. From the canonical Eq. (50),

$$\frac{dq_j}{dr} = H_{v_j}(r, q_j, q_\ell, v_j),$$

is seen that at corners, where a finite discontinuity in some or all of the controls  $q_\ell$  may occur, the states  $q_j$  are continuous but some or all of the slopes  $q_j'$  are discontinuous. In addition to Eqs. (48) to (50) the following relation may be obtained

$$\frac{d}{dr} H[r, q_j(r), q_\ell(r), v_j(r)] = H_r + H_{q_j} q_j' + H_{q_\ell} q_\ell' + H_{v_j} v_j'$$

which, accounting for Eqs. (48) to (50), leads to

$$\frac{dH}{dr} = H_r(r, q_j, q_\ell, v_j) \quad (52)$$

along the extremal arc.

Evidently, from Eq. (52), for  $H_r = 0$ ,  $H = \text{const.}$  and a first integral is obtained. Eq. (52) is the equivalent of Eq. (19), as is immediately verified using Eqs. (46) and (47). The canonical equations

$$\frac{dv_j}{dr} + H_{q_j}(r, q_j, q_\ell, v_j) = 0; \quad \frac{dH}{dr} - H_r(r, q_j, q_\ell, v_j) = 0 \quad (53)$$

indicate that at corners of sub-arcs satisfying Eqs. (53) the functions  $v_j$  are continuous as well as  $H$ , though  $H_{q_j}$  and  $H_r$  may be discontinuous. These are the Erdmann-Weierstrass vertex conditions derived in Section 1, Eqs. (24) and (25).

The canonical equations of the extremals, Eqs. (48), (49), (50) and (52) have been derived making use of the Legendre transformation

$$v_j = \Pi_{q_j'}, \quad v_j q_j' - \Pi = H$$

whose inverse is given by

$$H_{v_j} = q'_j, \quad v_j q'_j - H = \Pi$$

From Eqs. (48) to (50) the Euler equations derived in Section 1 may be readily obtained, using the relation  $v_j = \Pi_{q'_j}$  and Eqs. (47).

The transformation previously developed thus, leads to two equivalent problems for which the following statement applies: "Every extremal arc in the state variables  $q_j(r)$  and control variables  $q_\ell(r)$ ,  $r_1 \leq r \leq r_F$ , defines by means of the constraints  $\phi_i = 0$  and canonical variables  $v_j = \Pi_{q'_j}$  an admissible solution

$$q_j(r), q_\ell(r), v_j(r)$$

of the canonical equations

$$\frac{dq_j}{dr} = H_{v_j}, \quad \frac{dv_j}{dr} = -H_{q_j}, \quad H_{q_\ell} = 0.$$

Conversely, for every admissible solution of the canonical equations of the extremals, the state variables  $q_j(r)$  and control variables  $q_\ell(r)$  belong to an extremal".

#### 1. The Hamilton-Jacobi Equation and Caratheodory's Curves of Quickest Descent.

Some brief comments will be made now in connection with the Hamilton-Jacobi equation, canonical variables and the Carathéodory interpretation of the extremals as curves of quickest descent. Consider the hypersurface

$$W(r, q_j, q_\ell) = \rho = \text{const.}$$

Such hypersurface is said to be transversal to the direction  $1: q'_j$  if

$$(\Pi - q'_j \Pi_{q'_j}) dr + \Pi_{q'_j} dq_j = 0 \quad (54)$$

for any set  $dr, dq_j, dq_\ell$  satisfying

$$W_r dr + W_{q_j} dq_j + W_{q_\ell} dq_\ell = 0 \quad (55)$$

Now, in a field  $F$  of extremals, the Hilbert integral (Ref. 1)

$$J^* = \int_{r_0, q_{j_0}, q_{\ell_0}}^{r, q_j, q_\ell} \left[ \Pi - q'_j \Pi_{q'_j} \right] dr + \Pi_{q'_j} dq_j \quad (56)$$

formed with the slope functions of the field, is independent of the path. Then from a fixed point  $(r_0, q_j, q_{\ell_0})$  to a variable point  $(r, q_j, q_\ell)$  in  $F$ , the values of the integral in Eq. (56) define a single-valued function  $W(r, q_j, q_\ell)$  whose derivatives are

$$W_r = \Pi - q'_j \Pi_{q'_j} \quad (57)$$

$$W_{q_j} = \Pi_{q'_j} \quad (58)$$

$$W_{q_\ell} = 0 \quad (59)$$

The previous equations follow after considering that since the value  $J^*$  is independent of the path in  $F$ , then  $J^*$  must be the integral of an exact differential  $dW$ . Then, at any point  $(r, q_j, q_\ell)$  of the field and from Eqs. (54), (55), (57) to (59), the hypersurface  $W(r, q_j, q_\ell) = \rho = \text{const.}$  will cut transversally the extremal of the field passing through that point.

According to previous developments in Section 2 and from Eq. (58), it is now introduced the canonical variables  $(r, q_j, q_\ell, W_{q_j})$  related to the variables  $(r, q_j, q'_j, q_\ell, \mu_i)$  by the equations

$$W_{q_j} = \Pi_{q'_j}(r, q_j, q'_j, q_\ell, \mu_i), \quad 0 = \phi_i(r, q_j, q'_j, q_\ell) \quad (60)$$

By implicit function theorems (Ref. 1), there exists a neighborhood  $N$  of  $(r, q_j, q_\ell, W_{q_j}) \in F$  in which Eqs. (60) have solutions

$$q'_j = Q_j(r, q_j, q_\ell, W_{q_j}), \quad \mu_i = M_i(r, q_j, q_\ell, W_{q_j}) \quad (61)$$

reducing to  $q'_j(r), \mu_i(r)$  for  $[r, q_j(r), q_\ell(r), W_{q_j}(r)] \in F$ . Then with the canonical variables introduced the Hamiltonian functional, with the slopes and multipliers of the field, is written

$$H(r, q_j, q_\ell, W_{q_j}) = W_{q_j} Q_j - \Pi(r, q_j, q_\ell, Q_j, M_i) \quad (62)$$

Consequently, from Eqs. (57) and (62) is obtained

$$W_r + H(r, q_j, q_\ell, W_{q_j}) = 0 \quad (63)$$

Eq. (63) is the so-called Hamilton-Jacobi partial differential equation.<sup>\*</sup> The extremals are shown to be the characteristic curves of said partial differential equation of first order. Carathéodory (Refs. 10, 11) has studied geometric properties of the extremals, based on previous developments, leading to very interesting interpretations of variational problems. He identifies the extremals with what he calls "curves of quickest descent". For that, it is required that there exist a representation  $W(r, q_j, q_\ell) = \rho = \text{const}$ , (viz., a one-parameter family of hypersurfaces) for the family of transversal surfaces, with which  $W$  satisfies the Hamilton-Jacobi partial differential equation. Should such family exist, the curves of quickest descent are all extremals of the field  $F$ . In recent years, researches working in optimization problems have applied direct methods (e.g. Ref 12) based on ideas similar to those exposed in this paragraph.

The work of Carathéodory is intimately related to the theory of fields and their relationship with the partial differential equation of Hamilton and Jacobi. If the family  $W(r, q_j, q_\ell) = \rho$  exists, and satisfies Eq. (63), the curves of quickest descent are extremals and the surfaces form a family of so-called geodesically equidistant surfaces.

\* Equation introduced by Hamilton for problems of mechanics in 1835 (Ref. 8). The original introduction of canonical variables for variational problems associated with mechanics is attributed also to Hamilton, (Ref. 8), while Lagrange already used differential equations of canonical form in his theory of perturbations (Ref. 9).

### C. NECESSARY CONDITIONS FOR THE CASE OF RESTRICTED OR ONE-SIDED VARIATIONS

In previous sections the necessary conditions for a minimum of an arc  $q_j(r), q_\ell(r)$  have been analyzed tacitly assuming arbitrary admissible variations  $\delta q_j \geq 0$  and  $\delta q_\ell \geq 0$  on  $E$ . Some restrictions will be imposed now on the variations to the extent of studying the necessary conditions to be satisfied by a broken extremal arc  $E$  when a piece, or sub-arc of it, is in common with the boundaries of the region of admissible control.

Assume there exist a normal extremaloid  $E$  (i.e., of class  $D'$ , or piecewise  $C'$ ; Refs. 1 and 4) affording a relative minimum for the function  $\Lambda(q_{j_F}, q_{j_I}, r_F, r_I)$  in the class of admissible arcs

$q_j(r), q_\ell(r), r_I \leq r \leq r_F$ , satisfying the differential constraints  $\phi_i = 0$ , and end-constraints  $\psi_\rho = 0$ , and such that the control variable is  $q_\ell(r) = q_\ell(r)$ ,  $r_{c_1} \leq r \leq r_{c_2}$  where  $q_u(r) \geq q_\ell(r) \geq q_b(r)$ ,  $r_I \leq r \leq r_F$  defines a region  $D$  of admissible control.

The normal minimizing extremal  $E$  assumed is pictured in Figs. 4(a) and 4(b). The nomenclature and range of the subindices has been defined in previous sections, thus it will not be repeated.

Due to the "normality" property and assuming that  $\xi_I, \xi_F, \eta_j(r), \eta_\ell(r)$  is a set of admissible variations with which  $E$  satisfies the equations of variation  $\Phi_i(\eta_j, \eta_\ell, r) = 0$ ,  $\Psi_\rho(\xi_I, \xi_F, \eta_{j_I}, \eta_{j_F}) = 0$ , then from the Fundamental Imbedding Lemma (Ref. 1), a one-parameter family of arcs  $q_j(r, \epsilon), q_\ell(r, \epsilon)$  containing  $E$  for the value  $\epsilon = \epsilon_0$  and having  $\xi_I, \xi_F, \eta_j(r), \eta_\ell(r)$  as its variations on  $E$  can be determined. The varied functions of the family are

$$\begin{aligned} q_j(r, \epsilon) &= q_j(r, \epsilon_0) + (\epsilon - \epsilon_0) \eta_j(r) \\ q_\ell(r, \epsilon) &= q_\ell(r, \epsilon_0) + (\epsilon - \epsilon_0) \eta_\ell(r) \\ r_s(\epsilon) &= r_s(\epsilon_0) + (\epsilon - \epsilon_0) \xi_s, \quad s = I \text{ or } F \end{aligned} \tag{64}$$

and its variations on  $E(\epsilon = \epsilon_0)$  are then

$$\begin{aligned}
\left. \frac{\partial q_j(r, \epsilon)}{\partial \epsilon} \right|_{\epsilon = \epsilon_0} &= \eta_j(r) \\
\left. \frac{\partial q_\ell(r, \epsilon)}{\partial \epsilon} \right|_{\epsilon = \epsilon_0} &= \eta_\ell(r) \\
\left. \frac{dr_s(\epsilon)}{d\epsilon} \right|_{\epsilon = \epsilon_0} &= \xi_s, \quad s = I \text{ or } F
\end{aligned} \tag{65}$$

In the family (64),  $|\epsilon - \epsilon_0|$  is taken arbitrarily small, i.e.,  $\epsilon$  is taken so close to  $\epsilon_0$  that  $|\epsilon - \epsilon_0| \rightarrow |d\epsilon|$ . Thus, the  $\delta$ -variations are  $\delta q_j = (\epsilon - \epsilon_0) \eta_j$ ,  $\delta q_\ell = (\epsilon - \epsilon_0) \eta_\ell$

Forming the  $J$  sum in similar manner as was done in previous sections, and extending the integrals between corners [see Figs. 4(a) and 4(b)], the following derivative is found on  $F$

$$\begin{aligned}
\left. \frac{dJ(\epsilon)}{d\epsilon} \right|_{\epsilon = \epsilon_0} &= \left[ \lambda_0 (\Lambda_{q_F} q'_F + \Lambda_{r_F}) + \Pi_F + \lambda_\rho (\psi_{q_F} q'_F + \psi_{r_F}) \right] \xi_F \\
&+ \left[ \lambda_0 (\Lambda_{q_I} q'_I + \Lambda_{r_I}) - \Pi_I + \lambda_\rho (\psi_{q_I} q'_I + \psi_{r_I}) \right] \xi_I \\
&+ \left( \lambda_0 \Lambda_{q_F} + \Pi_{q'} \right)_F + \lambda_\rho \psi_{q_F} \eta_{j_F} + \left( \lambda_0 \Lambda_{q_I} - \Pi_{q'} \right)_I + \lambda_\rho \psi_{q_I} \eta_{j_I} \\
&+ \left( \Pi_{c_1-0} - \Pi_{c_1+0} \right) \xi_{c_1} + \left( \Pi_{c_2-0} - \Pi_{c_2+0} \right) \xi_{c_2} + \left( \Pi_{q', c_1-0} - \Pi_{q', c_1+0} \right) \eta_{j_{c_1}}
\end{aligned}$$

$$\begin{aligned}
& + (\Pi_{q_{c_2-0}} - \Pi_{q_{c_2+0}}) \eta_{j_{c_2}} - \int_{r_1}^{r_{c_1-0}} [\Pi]_{q_j} \eta_j dr \\
& + \int_{r_1}^{r_{c_1-0}} \Pi_{q_\ell} \eta_\ell dr - \int_{r_{c_1+0}}^{r_{c_2-0}} [\Pi]_{q_j} \eta_j dr + \int_{r_{c_1+0}}^{r_{c_2-0}} \Pi_{q_\ell} \eta_\ell dr \\
& - \int_{r_{c_2+0}}^{r_F} [\Pi]_{q_j} \eta_j dr + \int_{r_{c_2+0}}^{r_F} \Pi_{q_\ell} \eta_\ell dr
\end{aligned} \tag{66}$$

In taking the preceding derivative account is taken for the fact that the  $\eta_j$  variations satisfy similar continuity properties than the  $q_j$  functions. Then, at junctions of different sub-arcs the variations must be continuous, i.e.,  $\eta_j|_{c_1-0} = \eta_j|_{c_1+0}$ ,  $\eta_j|_{c_2-0} = \eta_j|_{c_2+0}$ ,

while some or all of the  $\eta_j'$  functions may have finite discontinuities. A graphical interpretation of the total differential

$dq_j \Big|_{\epsilon=\epsilon_0}$  at a corner C of E ( $\epsilon=\epsilon_0$ ) as pertaining to each sub-arc joining at C is given in Fig. 5. This figure takes account of the most general situation where E is imbedded in a class of admissible arcs  $q_j(r, \epsilon)$  whose corners C' define a "line of corners",  $q_j[r_c(\epsilon), \epsilon]$ ,  $r_c(\epsilon) = r_c(\epsilon_0) + (\epsilon - \epsilon_0)\xi_c$ , containing that of E for  $\epsilon = \epsilon_0$ .

From the transversality condition Eq. (12), the Erdmann-Weierstrass corner conditions Eq. (24), and since along each sub-arc of the minimizing extremal E where variations  $\delta q_j \geq 0$ ,  $\delta q_\ell \leq 0$  are admissible (viz., it may be taken  $\epsilon - \epsilon_0 \geq 0$ ) the Euler equations  $[\Pi]_{q_j} = 0$ ,  $[\Pi]_{q_\ell} = 0$ , (Sections 1 and 2), must be

satisfied, then the derivative (66) on E may be written

$$\left. \frac{dJ(\epsilon)}{d\epsilon} \right|_{\epsilon=\epsilon_0} = \int_{r_{c_1}+0}^{r_{c_2}-0} \Pi_{q_\ell} \eta_\ell dr. \quad (67)$$

Note now that due to the boundary  $q_{\ell_b}(r)$  imposed, [see Fig. 4(b)] the only admissible variations in control along the sub-arc  $(r_{c_1}, r_{c_2})$  are  $\delta q_\ell = (\epsilon - \epsilon_0) \eta_\ell \geq 0$ . Then for  $\eta_\ell(r, \epsilon_0) \geq 0$ ,

$r_{c_1} \leq r \leq r_{c_2}$ ,  $(\epsilon - \epsilon_0)$  must be taken positive. This means that, as shown in Fig. 4(d), now it can no longer be inferred that

$$\left. \frac{dJ(\epsilon)}{d\epsilon} \right|_{\epsilon=\epsilon_0} = 0 \quad \text{as is the case when } \epsilon - \epsilon_0 \text{ may be taken positive}$$

or negative [Fig. 4(c)] in a close neighborhood of the minimizing extremal  $E(\epsilon = \epsilon_0)$ . Since  $\epsilon - \epsilon_0$  may only be taken positive now, and E minimizes  $J(\epsilon)$  for  $\epsilon = \epsilon_0$ , then we may only infer that

$$\left. \frac{dJ(\epsilon)}{d\epsilon} \right|_{\epsilon=\epsilon_0} \geq 0, \quad [\text{Fig. 4(d)}]. \quad \text{To the left of } \epsilon = \epsilon_0, \text{ in Fig.}$$

4(d), there is a forbidden zone where, if values  $(\epsilon - \epsilon_0) < 0$  were permissible [i.e., if there were no boundary  $q_{\ell_b}(r)$ ]  $J(\epsilon)$  could yet attain a value  $J(\epsilon_p) < J(\epsilon_0)$  for a control

$q_\ell(r) = q_{\ell_b}(r) + \delta q_\ell(r)$ ,  $\delta q_\ell < 0$ ,  $r_{c_1} \leq r \leq r_{c_2}$ . Of course, to the right of

$\epsilon = \epsilon_0$  and in a close neighborhood  $\epsilon - \epsilon_0 \leq \rho$ ,  $\rho > 0$ , there is no admissible arc  $q_j(r, \epsilon)$ ,  $q_\ell(r, \epsilon)$  for which  $J(\epsilon) < J(\epsilon_0)$

because by hypothesis  $q_j(r, \epsilon_0)$ ,  $q_\ell(r, \epsilon_0)$  is a minimizing extremal (relative minimum). Thus, from the derivative (67) and previous discussion is obtained.

$$\Pi_{q_\ell} \geq 0, \quad \ell = m+1; \dots; s \quad (68)$$

as a necessary condition for a minimum associated with the control variables following the boundary of the region D of admissible control (i.e., for one-sided admissible control  $\delta q_\ell \geq 0$ ;  $q_{\ell_b}(\tau)$  is called a "lower" bound). If the admissible control variation is  $\delta q_\ell \leq 0$  the associated condition is similar to that in Eq. (68), but where the inequality sign is reversed, (e.g., case of "upper" control bound  $q_\ell(\tau) \leq q_{\ell_u}(\tau)$ ,  $\tau_I \leq \tau \leq \tau_F$ ).

In some cases, in correspondence with admissible control variations  $\delta q_\ell > 0$ ,  $\tau_{c_1} \leq \tau \leq \tau_{c_2}$ , only admissible state variations  $\delta q_j > 0$ , may be permissible. Then, similarly as before

$$[\Pi]_{q_j} \leq 0 \quad (69)$$

is obtained as a necessary condition for a minimum associated with state variables following a lower bound. Other cases may be considered in a similar manner. Condition (69) has been previously derived by Bolza (Ref. 4) and by Bliss and Underhill (Ref. 13). The admissible displacements must be carefully analyzed when applying the previous conditions to particular cases.

## D. PONTRYAGIN'S MAXIMUM PRINCIPLE AND THE WEIERSTRASS CONDITION

If an extremal arc  $E$  is locally compared with a differentiable admissible arc  $\mathcal{C}$  in the  $(r, q_j, q_\ell, v_j)$  - space, having with  $E$  projections of common coordinates  $(r, q_j, v_j)$  and such that for the control variable increment  $\Delta q_\ell = q_\ell - q_\ell^*$ , ( $q_\ell^*$  = optimum control on  $E$ ;  $q_\ell$  control on  $\mathcal{C}$ ) it is  $\Delta q'_j = q'_j(r, q_j, q_\ell, v_j) - q'_j(r, q_j, q_\ell^*, v_j) \neq 0$ ,  $\Delta q_j = 0$ , then at such point the necessary Weierstrass condition

$$W \equiv \Pi(r, q_j, q_\ell, v_j) - \Pi(r, q_j, q_\ell^*, v_j) - (q'_j - q_j'^*) v_j \geq 0 \quad (70)$$

must be satisfied. At  $(r, q_j, q_\ell^*, v_j) \in E$  correspond values,  $\mu_i = \bar{\mu}_i(r, q_j, q_\ell^*, v_j)$  and  $q'_j = \bar{q}'_j(r, q_j, q_\ell^*, v_j)$  of the multipliers and slopes of the field on  $E$ . A graphical representation of the increments  $\Delta v'_j \neq 0$ ,  $\Delta q'_j \neq 0$ ,  $\Delta v_j = 0$ ,  $\Delta q_j = 0$  with respect to the values on  $E$  and for a control increment  $\Delta q_\ell$  is shown in Figs. 6(a) and (b).

From Eqs. (45) or (46) and (70) we see that the Weierstrass condition in terms of the Hamiltonian  $H$  or, in other words, the canonical expression of the Weierstrass condition may be written as

$$\Delta H = H(r, q_j, q_\ell, v_j) - H(r, q_j, q_\ell^*, v_j) \leq 0 \quad (71)$$

for any admissible  $q_\ell$ . Eq. (71) must be satisfied at any point  $(r, q_j, q_\ell^*, v_j)$  on the extremal  $E$  i.e., at any point along an admissible solution of the canonical equations of the extremals.

Assume now that the minimizing extremal arc  $E$  (relative minimum), represented in Figs. 4(a) and (b), is expressed in terms of the canonical variables  $q_j(r), q_\ell(r), v_j(r)$ . Then, following an analogous derivation and with similar considerations as were made in Section 3, it can be shown that for one-sided variations  $\delta q_\ell = \eta_\ell(r, \epsilon_0)(\epsilon - \epsilon_0) \geq 0$ ,  $\eta_\ell(r) \geq 0$ ,  $r_{c_1+0} \leq r \leq r_{c_2-0}$ ,  $(\epsilon - \epsilon_0) > 0$ ,  $|\epsilon - \epsilon_0| \rightarrow |d\epsilon|$ , we obtain the necessary condition

$$dJ(\epsilon) \Big|_{\epsilon = \epsilon_0} = \int_{\tau_{c_1} + 0}^{\tau_{c_2} - 0} F_{q_\ell} \delta q_\ell d\tau = - \int_{\tau_{c_1} + 0}^{\tau_{c_2} - 0} H_{q_\ell} \delta q_\ell d\tau \geq 0 \quad (72)$$

for  $J(\epsilon_0)$  to be a minimum with respect to any other value  $J(\epsilon)$ ,  $0 < (\epsilon - \epsilon_0) < \rho$ , and with  $\rho > 0$  defining an arbitrarily close neighborhood of  $\epsilon_0$ . Consequently, Eq. (72) requires

$$H_{q_\ell} [r, q_j(r), q_\ell(r), v_j(r)] \leq 0 \quad (73)$$

at any time  $\tau, \tau_{c_1} + 0 \leq \tau \leq \tau_{c_2} - 0$ , and for  $\delta q_\ell(r, \epsilon_0) \geq 0$ , as a necessary condition for a minimum along sub-arcs where the control  $q_\ell(r)$  is in common with a lower bound  $q_\ell(r) = q_{\ell_b}(r)$  of the admissible domain  $D$  of control. Similarly,

$$H_{q_\ell} [r, q_j(r), q_\ell(r), v_j(r)] \geq 0 \quad (74)$$

for  $\delta q_\ell(r, \epsilon_0) \leq 0$ , is a necessary condition for a minimum along sub-arcs where the control  $q_\ell(r)$  is in common with an upper bound  $[q_\ell(r) = q_{\ell_u}(r)]$  of the admissible domain  $D$  of control.

In case the control  $q_\ell(r)$  belongs to the interior of  $D$  [Fig. 4(b), sub-arcs  $IC'_1$  and  $C'_2 F$ ] then control variations  $\delta q_\ell \geq 0$  are admissible and Eq. (49) must hold.

Note that in the previous considerations the most general situation of time-dependent control boundaries has been assumed. It is then concluded that

$$\frac{\partial H(r, q_j, q_\ell, v_j)}{\partial q_\ell} = 0 \text{ for admissible } \delta q_\ell \geq 0$$

$$\frac{\partial H(r, q_j, q_\ell, v_j)}{\partial q_\ell} \geq 0 \text{ for admissible } \delta q_\ell \leq 0$$

$$\frac{\partial H(r, q_j, q_\ell, v_j)}{\partial q_\ell} \leq 0 \text{ for admissible } \delta q_\ell \geq 0 \quad (75)$$

$$\Delta H = H(r, q_j, q_\ell^* + \Delta q_\ell, v_j) - H(r, q_j, q_\ell^*, v_j) \leq 0 \quad (76)$$

$q_\ell^*$  = control on  $E$ , at any given time  $r = r^*$ ,  $r_1 \leq r^* \leq r_F$ , with  $(r, q_j, v_j) \in E$ , constitute a set of necessary and sufficient conditions defining the optimum control  $q_\ell = q_\ell^*$ , in the bounded interval  $q_{\ell_b}(r^*) \leq q_\ell \leq q_{\ell_u}(r^*)$ , as the  $q_\ell$ -value maximizing  $H(r, q_j, q_\ell, v_j)$ . Note that the concept of "point by point variation", which as stated by Bolza (Ref. 4), played an important role in the early studies of the calculus of variations, is applied here. In fact, at a given value  $r = r^*$ ,  $\delta q_\ell = \delta q_\ell(r^*, \epsilon_0)$ , is the admissible variation  $\delta q_\ell$  at a "point" of  $q_\ell^*(r)$  on  $E$  [see Fig. 4(b) point M]. The maximizing property of the optimum control  $q_\ell^*$  in connection with the Hamiltonian  $H(r, q_j, q_\ell, v_j)$  is also called "Pontryagin's Maximum Principle" [Eq. (76)]. In terms of this maximality the optimum control at  $r = r^*$  on  $E$  is also expressed by  $[q_j(r^*) = q_j^*; v_j(r^*) = v_j^*]$

$$\begin{aligned} & \text{Max.} \quad H(r^*, q_j^*, v_j^*, q_\ell) \\ & q_{\ell_b}(r^*) \leq q_\ell \leq q_{\ell_u}(r^*) \end{aligned}$$

Note that for a normal extremal  $E$ , at a junction ( $r = r_c$ ) of two extremal sub-arcs, from Eq. (52) is obtained [e.g., see Fig. 4(b) at the corner  $r = r_{c1}$ ]

$$H[r_c, q_j(r_c), q_\ell^*(r_c - 0), v_j(r_c)] - H[r_c, q_j(r_c), q_\ell^*(r_c + 0), v_j(r_c)] = 0 \quad (77)$$

and thus, from Eqs. (45) or (46), (70) and (77), follows that the Weierstrass condition

$$W(r_c, q_{jc}, v_{jc}, q_{\ell_{c-0}}, q_{\ell_{c+0}}) = 0 \quad (78)$$

at the corner  $C$  (and consequently also  $\Delta H|_C = 0$ ). Eq. (78) states that at junctions of extremal sub-arcs, forming the extremal arc  $E$ , the Weierstrass condition vanishes. A graphical interpretation of the important condition given by Eq. (77), at junctions of extremal sub-arcs, is shown in Fig. 7(b) where some possible situations that may arise are pictured. Fig. 7(a) represents the optimum control identified by points of the  $H$ -line satisfying the conditions given by Eqs. (75) and (76).

The relation between Pontryagin's Maximum Principle and the Weierstrass condition has been also considered in Refs. 14, 16 and 17. The drawings corresponding to conditions (75) and (76) have been simplified in Fig. 7(a) adopting a two-dimensional representation. However, it is easy to imagine the corresponding condition represented in higher dimensional spaces (case of multiple control). Assuming two control variables,  $q_{m+1}$  and  $q_{m+2}$ , for example, the optimum controls  $q_{m+1}^*$  and  $q_{m+2}^*$  at given values  $(r, q, v_j) \in E$ , are the coordinates of the maximum point on the surface  $\Pi = \Pi(r, q_j, v_j, q_{m+1}, q_{m+2})$  in the interior or on the boundaries of the domain  $D$  of admissible control (see Fig. 8).

# E. PROBLEMS WITH PARTICULAR FORM OF THE DIFFERENTIAL CONSTRAINTS

The Mayer problem formulated in Sections 1 and 2, will be now considered in the class of arcs

$$q_j(r), \dots, q_\ell(r), \quad r_1 \leq r \leq r_F, \quad j = 1, \dots, m$$

satisfying differential constraints of the form

$$\phi_j \equiv q'_j - f_j(q_1, \dots, q_m, q_\ell, r) = 0, \quad j = 1; 2; \dots; m \quad (80)$$

where  $q_\ell$  is a control variable. The terminal conditions are

$$\psi_\rho [q_j(r_F), q_j(r_1), r_F, r_1] = 0, \quad \rho = 1, \dots, r \leq 2m + 1 \quad (81)$$

and the function to be minimized is

$$\Lambda = \Lambda [q_j(r_F), q_j(r_1), r_F, r_1]$$

The control variable  $q_\ell(r)$  may or may not be bounded  $[q_{\ell_h}(r) \leq q_\ell(r) \leq q_{\ell_u}(r)]$ . This will be opportunely considered in the following discussion. Also, it will be assumed that

$$f_1(q_j, q_\ell, r) = \varphi_1(q_j, q_\ell, r), \quad f_2(q_j, q_\ell, r) = \varphi_2^-(q_j, r) q_\ell$$

and

$$\frac{\partial f_s}{\partial q_\ell} = 0, \quad s = 3; \dots; m$$

With the preceding assumptions a mathematical model representing the type of constraints most generally found in problems of the mechanics of flight of airplanes, glide vehicles, terrestrial and extra-terrestrial rocket propelled vehicles, has been defined. Extensions of the following considerations to cases of multiple control will be considered "in situ". For the variational problem of minimizing  $\Lambda(q_{j_F}, q_{j_1}, r_F, r_1)$  in the class of arcs  $q_j(r), q_\ell(r)$  satisfying constraints (80) and (81), similar necessary conditions as previously derived in Sections 1, 2 and 3 apply. Thus, this will not be done again here; it is a simple matter only requiring easy specializations of previous developments. The same applies, of course, in regard

to imbedding considerations, continuity and differentiability conditions. The previous general analysis undertaken in Sections 1 to 5 will be here particularly applied, with the object of presenting to the reader the "characteristic line" construction (Zermelo's line, or so-called "indicatrix line" by Cicala, Ref. 3) and some other geometrical interpretations of interest for problems with bounded and unbounded control.

Forming the Euler-Lagrange sum  $\Pi = \mu_j \phi_j$  and from Eqs. (80) and (42), it is seen that now the canonical variables  $v_j = \Pi_{q_j}$  are identical with the multipliers  $\mu_j$ . Thus, for the type of differential constraints assumed, the canonical transformation (see Section 2) degenerates into a trivial form. This fact, however, does not impair the validity of the application of previous considerations to the case studied here. Therefore, the Legendre transformation of the variational problem (see Section 2) will be formally introduced and the canonical notation kept for the sake of referring and applying previous conditions derived using the generalized canonical form.

The canonical variables  $(r, q_j, q_j', v_j)$  are related to the variables  $(r, q_j, q_j', q_\ell, \mu_j)$  by the equations

$$v_j = \Pi_{q_j'} = \mu_j, \quad 0 = q_j' - f_j(q_1, \dots, q_m, q_\ell, r) \quad (82)$$

The properties of these variables, (continuity, etc) may be derived from considerations in the preceding sections. Eq. (82), in a neighborhood  $N$  of  $(r, q_j, q_\ell, v_j) \in E$  have solutions

$$q_j' = \bar{q}_j'(q_1, \dots, q_m, q_\ell, v_j, r) = \bar{f}_j(q_1, \dots, q_m, q_\ell, r), \quad \mu_j = \bar{\mu}_j(q_1, \dots, q_m, q_\ell, r, v_j) = \bar{v}_j \quad (83)$$

reducing to  $q_j'(r), \mu_j(r)$ , for  $[q_j(r), q_\ell(r), v_j(r), r] \in E$ . Following similar developments as indicated in Eqs. (44) to (47) it is found that the equations of variation, in terms of the canonical variables introduced, are

$$[F]_{q_j} = \frac{dv_j}{dr} + H_{q_j}(r, q_j, q_\ell, v_j) = 0 \quad (84)$$

$$[F]_{v_j} \equiv \frac{dq_j}{dr} - H_{v_j}(r, q_j, q_\ell, v_j) = 0 \quad (85)$$

The previous equations, using Eqs. (47) and (82), are readily reduced to the corresponding differential set, (equations of constraint and Euler equations of the Mayer Problem)

$$\frac{d\mu_j}{dr} + \mu_i \frac{\partial f_i}{\partial q_j} = 0, \quad j = 1; \dots; m, \quad i = 1, \dots, m \quad (86)$$

$$\frac{dq_j}{dr} - f_j(q_1, \dots, q_m, q_\ell, r) = 0, \quad (87)$$

expressed in the  $(r, q_j, q'_j, q_\ell, \mu_j)$  variables. Eqs. (84) to (87), giving the first necessary conditions for an extremal in the  $(r, q_j, q_\ell, v_j)$  and  $(r, q_j, q'_j, q_\ell, \mu_j)$  systems, (except for one equation to be considered in the following), evidently correspond with Eqs. (6), (13), (14) and (15) of Boltyanskii, Gamkrelidze and Pontryagin in Ref. 15. According to previous considerations in Sections 2, 3 and 4 the following equation, associated with the variation of the control variable  $q_\ell(r)$ , must be added to Eqs. (84) and (85), [see Eqs. (49), (73) and (74)]

$$\frac{\partial H}{\partial q_\ell} = v_j \frac{\partial f_j}{\partial q_\ell} = 0, \quad \text{for admissible } \delta q_\ell \gtrless 0 \text{ on } E \quad (88a)$$

$$\frac{\partial H}{\partial q_\ell} = v_j \frac{\partial f_j}{\partial q_\ell} \leq 0, \quad \text{for admissible } \delta q_\ell \geq 0 \text{ on } E \quad (88b)$$

$$\frac{\partial H}{\partial q_\ell} = v_j \frac{\partial f_j}{\partial q_\ell} \geq 0, \quad \text{for admissible } \delta q_\ell \leq 0 \text{ on } E. \quad (88c)$$

Thus, the canonical differential equations of the extremals given by Eqs. (84), (85) and either one among Eqs. (88a), (88b) and (88c), (according to the permissible control variations on the extremal), define a determined system of  $2m+1$  differential equations in the  $2m+1$  variables  $q_j(r), v_j(r), q_\ell(r)$ . Because of similar reasons, either one of the equations (see Section 3)

$$\frac{\partial \Pi}{\partial q_\ell} = -\mu_j \frac{\partial f_j}{\partial q_\ell} = 0, \quad \text{for admissible } \delta q_\ell \gtrless 0 \text{ on } E \quad (88d)$$

$$\frac{\partial \Pi}{\partial q_\ell} = - \mu_j \frac{\partial f_j}{\partial q_\ell} \geq 0, \text{ for admissible } \delta q_\ell \geq 0 \text{ on } E \quad (88e)$$

$$\frac{\partial \Pi}{\partial q_\ell} = - \mu_j \frac{\partial f_j}{\partial q_\ell} \leq 0, \text{ for admissible } \delta q_\ell \leq 0 \text{ on } E \quad (88f)$$

must be added to the system of Eqs. (86) and (87), thus obtaining a set of  $2m+1$  differential equations in the  $2m+1$  variables  $q_j(r)$ ,  $\mu_j(r)$ ,  $q_\ell(r)$ . Incidentally, note that the Legendre transformation of the Mayer problem in Section 2 (and consequently also here) into the canonical form is of an involutory character in as much as the constraints of one problem go over into the natural conditions of the other.

The transformation introduced in the present case [Eq. (82)] does not operate in reality any "mapping" between the regions N and M defined in Section 2 (see Fig. 3) since any interior point,  $P'(r, q_j, q_\ell, v_j) \equiv P(r, q_j, q'_j, q_\ell, \mu_j)$ , (the one-to-one correspondence is transformed into an identity).

#### 1. The Characteristic Line, the H-Line and Their Properties.

For the specific set of constraints [Eq. (80)] being considered, the Weierstrass condition is written [Section 4, Eq. (70)]

$$W \equiv - \left[ f_j(q_1, \dots, q_m, q_\ell^* + \Delta q_\ell, r) - f_j(q_1, \dots, q_m, q_\ell^*, r) \right] v_j \geq 0 \quad (89)$$

since  $\Pi$  vanishes identically along any admissible arc. Eq. (89), accounting for the particular form assumed for the functions  $f_j(r, q_1, \dots, q_m, q_\ell)$  in Section 6, is now rewritten as

$$W(r, q_j, q_\ell, q_\ell^*, v_j) \equiv - \Delta \varphi_1 v_1 - \varphi_2(q_j, r) \Delta q_\ell v_2 \geq 0 \quad (90)$$

where

$$\Delta \varphi_1 = \varphi_1(q_1, \dots, q_m, q_\ell^* + \Delta q_\ell, r) - \varphi_1(q_1, \dots, q_m, q_\ell^*, r)$$

Similarly, from Eq. (79), the Weierstrass condition may be written in the variables  $(r, q_j, q'_j, q_\ell, \mu_j)$  as

$$W(r, q_j, q_\ell, q_\ell^*, q'_j, \mu_j) \equiv - \Delta \varphi_1 \mu_1 - \varphi_2(q_j, r) \Delta q_\ell \mu_2 \geq 0 \quad (91)$$

The Hamiltonian  $H$  is

$$\begin{aligned} H(r, q_j, q_\ell, v_j) &= v_j f_j(q_1, \dots, q_m, q_\ell, r) = \\ &= v_1 \varphi_1(q_j, q_\ell, r) + v_2 \varphi_2(q_j, r) q_\ell + \dots + v_m f_m(q_j, r) \end{aligned} \quad (92)$$

From Eqs. (39) and (40) is seen that the Legendre-Clebsch condition is now expressed by

$$\Pi_{q_\ell q_\ell}(\mu_j, r, q_j, q'_j, q_\ell) (\delta q_\ell)^2 = -\mu_1 \frac{\partial^2 \varphi_1(q_j, q_\ell, r)}{\partial q_\ell^2} (\delta q_\ell)^2 \geq 0 \quad (93)$$

at any point  $(r, q_j, q'_j, q_\ell, \mu_j)$  on the extremal and for any admissible  $\delta q_\ell(r)$ . Finally, from Eqs. (90) and (92), it is clearly seen that the canonical expression of the Weierstrass condition leads to the Maximum Principle

$$\Delta H = H(r, q_j, q_\ell, v_j) - H(r, q_j, q_\ell^*, v_j) = -W(r, q_j, q_\ell, q_\ell^*, v_j) \leq 0 \quad (94)$$

for any given set  $(r, q_j, q_\ell^*, v_j)$  belonging to an extremal of the Mayer field.

The function  $\varphi_1(q_j, q_\ell, r)$  will be called "characteristic function". For a given set  $(r, q_j) \in E$ , the function  $\varphi_1$  in terms of  $q_\ell$  defines a "characteristic line", which for each admissible given value of the control in the interval  $q_b(r) \leq q_\ell(r) \leq q_u(r)$  defines a corresponding "characteristic value". The geometrical interpretation of the relation between the Weierstrass condition and the Legendre-Clebsch condition with analogous lines was first introduced by Zermelo (see, e.g., Ref. 19), for problems of the Lagrangian form. Then Cicala in recent years (e.g., see Ref. 3) applied a similar device to represent geometrically said necessary conditions for problems of the Mayer form. Cicala called "indicatrix line" the lines which the geometrical construction was based on. Incidentally, this name was also used previously by Carathéodory (Ref. 20) to designate with a similar construction of particular interest in studying broken extremals and necessary conditions.

The characteristic lines here applied correspond with that used by Cicala (indicatrix line). Let us assume, for the sake of discussion, that at any given values  $(r, q_j)$  the curvature of the characteristic line  $\varphi_1(r, q_j, q_\ell)$  is such that

$$\frac{\partial^2 \varphi_1(q_j, q_\ell, r)}{\partial q_\ell^2} < 0 \quad (95)$$

for any admissible control. Therefore, assuming that along the extremal,  $\mu_1(r) > 0$ ,  $r_1 \leq r \leq r_F$ , the Legendre condition (93) will be satisfied in its strengthened form at any point  $(r, q_j, q'_j, q_\ell, \mu_j) \in E$ . This assumption is a necessary requisite for the analysis of accessory problems (Ref. 1), as well as for the geometrical interpretation of the necessary conditions here studied and its justification will become apparent to the reader later on. First considered will be the case of unbounded control  $q_\ell(r)$ . Thus, this implies that Eq. (88d) must be in force, since for any  $q_\ell^*$  on the extremal,  $\delta q_\ell \geq 0$  will be admissible. Consequently, from Eqs. (80) and (88d) is derived

$$\mu_1 \frac{\partial \varphi_1}{\partial q_\ell} + \mu_2 \varphi_2(r, q_j) = 0, \quad (96)$$

as the necessary condition associated with the variation of the control variable. From Eq. (91) and the requirement  $\mu_1 > 0$  is seen that the Weierstrass condition may be rewritten as

$$\frac{W}{\mu_1} = -(\Delta \varphi_1 + \mu^* \Delta q_\ell) \geq 0 \quad (97)$$

where  $\mu^* = \frac{\mu_2}{\mu_1} \varphi_2(r, q_j)$  is Cicala's "index-value" (Ref. 3).

Eqs. (96) and (97) permit a graphical representation of the Weierstrass condition as indicated in Figure 10. Note that at any point on the characteristic line (e.g., P or Q) due to conditions (95) and  $\mu_1 > 0$ , the Weierstrass condition Eq. (97) is satisfied in its strengthened form ( $W/\mu_1 > 0$ ) for any admissible control increment  $\Delta q_\ell \geq 0$ . At point P in Figure 10 the Weierstrass condition in Eq. (91) is written

$$W(r, q_j, q_{\ell_P}, q_{\ell_S}, \mu_j) \equiv -\Delta \varphi_1 \mu_{1P} - \varphi_2 \Delta q_\ell \mu_{2P} > 0 \quad (98)$$

for a control increment  $\Delta q_\ell = q_{\ell_S} - q_{\ell_P} < 0$  and  $\Delta \varphi_1 = \varphi_{1_S} - \varphi_{1_P}$ . From Eq. (80), the slopes  $q'_{jP} = f_j(q_1, \dots, q_m, q_\ell, r)|_P$  may be determined. Also, from the angular coefficient of the tangent to the characteristic line at P the index-value  $\mu_P^* = \mu_2/\mu_1 \varphi_2|_P$  and therefore the ratio of multipliers  $\mu_2/\mu_1|_P$  may be found. Consequently, since there are

values  $(r_p, q_{jp}, q'_{jp}, q_{\ell p}, \mu_{jp})$  satisfying the Weierstrass condition at P for any admissible increment  $\Delta q_{\ell}$ , we may determine the values  $v_{jp}$  with which at  $(r_p, q_{jp})$  Eq. (94) is satisfied for a control  $q_{\ell} = q_{\ell p}$ . In our case, due to the simple canonical transformation applied evidently the previous statement is readily verified from Eqs. (90) to (92).

Note that since  $\mu_1 > 0$  then  $v_1 > 0$  and the modified Hamiltonian [from Eq. (92)]

$$\frac{H}{v_1} = \bar{H} = \varphi_1 + \bar{v}_2 \varphi_2 q_{\ell} + \dots + \bar{v}_m f_m \quad (99)$$

in terms of the ratio of canonical variables

$$\bar{v}_2 = \frac{v_2}{v_1}, \dots, \bar{v}_m = \frac{v_m}{v_1},$$

may be used in our reasoning. In this case then

$$\Delta \bar{H} = - \frac{W}{\mu_1} \leq 0 \quad (100)$$

is the necessary requirement for optimum control, equivalent to Eq. (94).

Corresponding to  $(r, q_j, q'_j, q_{\ell}, \mu_j)|_P$  there is, according to previous considerations, a point  $(r, q_j, q_{\ell}, v_j)|_P$  satisfying with this set of values the Weierstrass condition Eq. (90) and thus Eq. (94). The control variable  $q_{\ell p}$  may be therefore determined by drawing the H-line [Eq. (92)] for given  $(r, q_j, v_j)|_P$  and finding the control  $q_{\ell p}$  for which  $\Delta H = H(r, q_j, q_{\ell}, v_j) - H(r, q_j, q_{\ell p}, v_j) \leq 0$ . The graphical representation is shown in Figure 9. At point P in Figure 9 evidently  $H_{q_{\ell}} = 0$  since variations  $\delta q_{\ell} \gtrless 0$  are admissible.

Figures 9 and 10 point out certain basic geometrical characteristics. While the maximality principle [Figure 9] refers to function values, the Weierstrass device [Figure 10] emphasizes on function curvature. Thus the latter construction requires that the characteristic line lies entirely on the same side with respect to the slope at a given point, in order that the W-test be satisfied for any increment  $\Delta q_{\ell}$ . Moreover, there must not be

any inflexion point on the characteristic line otherwise in some interval the Weierstrass condition will not be satisfied. This follows from Eq. (93) since a reversal of the inequality

$$\frac{\partial^2 \varphi_1}{\partial q_\ell^2} > 0 \quad \left[ \frac{\partial^2 \varphi_1}{\partial q_\ell^2} = 0, \quad \text{inflexion point} \right]$$

will cause that the Legendre condition will no longer be satisfied with  $\mu_1 > 0$ .

In this respect an interesting case, though perhaps of purely theoretical value, may be analyzed. Consider, as shown in Fig. 11(b), that the characteristic line has the two asymptotes indicated with  $m$  and  $n$  and the two inflexion points  $M$  and  $N$ . One can immediately see that the Weierstrass condition then will not be satisfied at any point  $P$  on the characteristic for arbitrarily chosen  $\Delta q_\ell$ . However, if the control is now permitted to be "naturally bounded", viz.,  $q_{\ell M} \leq q_\ell \leq q_{\ell N}$ , then within the specified interval the W-test is satisfied for admissible values  $\Delta q_\ell$ . To  $(r, q_j, q'_j, q_\ell, \mu_j) \big|_P$  corresponds a canonical set satisfying, for given  $(r, q_j, v_j) \big|_P$ , the condition  $\Delta H \leq 0$  with  $q_\ell = q_{\ell P}$  and for any admissible  $\Delta q_\ell$ ,  $q_{\ell M} \leq q_\ell \leq q_{\ell N}$ . The H-line representing the latter condition at  $P$  is shown in Fig. 11(a). As derived from Figs 11(a) and 11(b), we could extend in many detailed considerations in regard to the configuration of the lines, characteristics, piece-wise admissibility, etc. However, for brevity, this will be left to the reader. The problem considered is of theoretical, and perhaps practical, value and may be synthesized in that class of "problems which having no optimum solution for unbounded control may be transformed into another having an optimum solution for naturally bounded control".

Another interesting case, which will be only mentioned here is that of a characteristic line having one inflexion point and only one asymptote.

It is interesting to point out that the considerations already made, concerning the graphical interpretation of the Weierstrass condition and maximality condition, may be readily extended to cases of multiple control. To show this we consider Eq. (80) to be of the following form

$$\phi_j \equiv q_j' - f_j(q_1, \dots, q_m, q_{\ell_1}, q_{\ell_2}, \tau) = 0 \quad (101)$$

where  $q_{\ell_1}(\tau)$  and  $q_{\ell_2}(\tau)$ ,  $\tau_1 \leq \tau \leq \tau_F$ , are now two control variables. It is assumed that

$$f_1 = \varphi_1(q_j, q_{\ell_1}, q_{\ell_2}, \tau) \quad ; \quad f_2 = \varphi_2(q_j, \tau) q_{\ell_1} \quad ; \quad f_3 = \varphi_3(q_j, \tau) q_{\ell_2}$$

$$\frac{\partial f_s}{\partial q_{\ell_1}} = \frac{\partial f_s}{\partial q_{\ell_2}} = 0 \quad ; \quad s = 4; \dots; m$$

For unbounded control the first necessary conditions, Eqs. (84) or (86) remain formally the same while the equations associated with the variation of the control variables are  $\Pi_{q_{\ell_1}} = H_{q_{\ell_2}} = 0$  or  $\Pi_{q_{\ell_1}} = \Pi_{q_{\ell_2}} = 0$ .

From the latter equations is obtained

$$\begin{aligned} \mu_1 \frac{\partial \varphi_1}{\partial q_{\ell_1}} + \mu_2 \varphi_2(q_j, \tau) &= 0 \\ \mu_1 \frac{\partial \varphi_1}{\partial q_{\ell_2}} + \mu_3 \varphi_3(q_j, \tau) &= 0 \end{aligned} \quad (102)$$

Also, consider that the Legendre condition

$$- \mu_1 \left[ \frac{\partial^2 \varphi_1}{\partial q_{\ell_1}^2} (\delta q_{\ell_1})^2 + 2 \frac{\partial^2 \varphi_1}{\partial q_{\ell_1} \partial q_{\ell_2}} \delta q_{\ell_1} \delta q_{\ell_2} + \frac{\partial^2 \varphi_1}{\partial q_{\ell_2}^2} (\delta q_{\ell_2})^2 \right] > 0 \quad (103)$$

is satisfied in the preceding strengthened form with  $\mu_1(\tau) > 0$ ,  $\tau_1 \leq \tau \leq \tau_F$ . Thus the bracketed term in Eq. (103) is less than zero for any admissible set of variations  $\delta q_{\ell_1}, \delta q_{\ell_2}$  satisfying  $\Phi_j(\delta q_j', \delta q_{\ell_1}, \delta q_{\ell_2}, \tau) = 0$ .

The Weierstrass condition is written

$$W = - \mu_1 \left[ \Delta \varphi_1 + \mu^* \Delta q_{\ell_1} + \bar{\mu}^* \Delta q_{\ell_2} \right] \geq 0 \quad (104)$$

where

$$\Delta \varphi_1 = \varphi_1(q_j, q_{\ell_1}^* + \Delta q_{\ell_1}, q_{\ell_2}^* + \Delta q_{\ell_2}, \tau) - \varphi_1(q_j, q_{\ell_1}^*, q_{\ell_2}^*, \tau)$$

and  $(q_{\ell_1}^*, q_{\ell_2}^*)$  indicate the control on the extremal at time  $\tau$ .

In condition (104), the values

$$\mu^* = \frac{\mu_2 \varphi_2(q_j, \tau)}{\mu_1}, \quad \tilde{\mu}^* = \frac{\mu_3 \varphi_3(q_j, \tau)}{\mu_1} \quad (105)$$

are the "index-values" on the extremal.

From previous considerations it is seen that the Weierstrass condition admits now a graphical interpretation as shown in Fig. 12. The characteristic line of Fig. 10(b) and Fig. 11(b) is now replaced by a "characteristic surface". To satisfy Eq. (104) for any set of control increments  $(\Delta q_{\ell_1}, \Delta q_{\ell_2})$  such that  $q_{\ell_1} = q_{\ell_1}^* + \Delta q_{\ell_1}$ ,  $q_{\ell_2} = q_{\ell_2}^* + \Delta q_{\ell_2}$ , are admissible [viz., each point  $P(q_{\ell_1}, q_{\ell_2})$  belongs to the interior or the boundaries of the region  $D$  of admissible control in Fig. 12] is necessary that the characteristic surface be entirely in one side with respect to the tangent plane  $T$  to the surface at any point  $P$  considered on it. In Fig. 12 is represented the case where every point on the surface, corresponding to controls  $q_{\ell_1}, q_{\ell_2}$ , in the admissible domain  $D$ , satisfy Eq. (104). At the point  $P$  on the characteristic surface, the geometrical representation may be reduced to the two-dimensional cases already studied by assuming admissible control increments  $\Delta q_{\ell_1} = 0$  (plane  $q_{\ell_1} = \text{const.}$ ),  $\Delta q_{\ell_2} \neq 0$  or viceversa.

Again as before, several cases of characteristic surface configurations may be readily imagined. This will not be discussed here.

Finally, note that with similar considerations as before we can see that the corresponding condition

$$\Delta H = H(r, q_j, q_{\ell_1}, q_{\ell_2}, v_j) - H(r, q_j, q_{\ell_1}^P, q_{\ell_2}^P, v_j) \leq 0, \text{ for } (r, q_j, v_j)$$

at  $P$  is represented by the point of maximum  $H$  on the surface  $H(r, q_j, v_j, q_{\ell_1}, q_{\ell_2})$  as was already shown in Fig. 8.

## 2. The Characteristic Line and the H-Line for the Case of Bounded Control.

For the case of bounded control the characteristic line and the H-line remain the same as for the case of unbounded control.

The Weierstrass condition still applies when the control is on the boundary, (i.e., for one-sided admissible variations  $\delta q$ ), as well as the maximality condition. This fact has been also pointed out in Ref. 17 by Kopp and previously by Cicala in Ref. 3. However, in Section 8 of Ref. 15 is stated that for boundary points of the admissible region of control the Weierstrass condition  $W \geq 0$ , in general, cease to be valid. Thus, this point will be reviewed here in some basic aspects and a geometrical interpretation will be presented (see also Refs. 3 and 17). Here, it will be again considered the case of one control variable  $q(r)$  for simplicity of presentation. Along a sub-arc of the extremal (see Fig. 4) where the control variable is in common with the boundary, of course, Eq. (88a) or the equivalent Eq. (88d) must be abandoned. Assume the control variable follows a time-dependent upper bound  $q_{\max}(r)$ . Then, at any time  $r$  on said sub-arc, the admissible control variation is  $\delta q \leq 0$  and therefore Eqs. (88c) or (88f) must be satisfied. The characteristic line is now drawn as indicated in Fig. 13(b). For a control value  $q = q_{\max}$ , Eq. (88f) requires

$$\mu^* \geq - \left( \frac{\partial \varphi_1}{\partial q} \right) \quad (106)$$

where  $\mu^*$  is the index-value  $\frac{\mu_2 \varphi_2(q_j, r)}{\mu_1}$ , ( $\mu_1 > 0$ ), then

this indicates that at R [Fig. 13 (b)] instead of referring the Weierstrass condition with respect to the slope  $n$  it must now be used the line  $n'$  of angular coefficient  $-\mu^*$ . From the Weierstrass condition Eq. (97) and Fig. 13(b), it is seen that then, for any  $\Delta q < 0$  with respect to  $q = q_{\max}$ , the Weierstrass condition is satisfied and is proportional to the segment between the characteristic line and the line  $n'$ . The arrows pointing upwards in Fig. 13(b) indicate positive values of the Weierstrass condition. From the drawing in Fig. 13(b) we may also clearly see that, if the Weierstrass condition is satisfied for a value

$$\mu_R^* = - \left. \frac{\partial \varphi_1}{\partial q} \right|_R,$$

as required for the case of unbounded control, it surely will be satisfied for

$$\mu_R^* \geq - \left. \frac{\partial \varphi_1}{\partial q} \right|_R$$

as the case of bounded control requires.

Following similar considerations the index-value  $\mu^* \leq \frac{\partial \varphi_1}{\partial q_\ell}$  indicates, for the case when the control is in common with a time-dependent (or fixed) lower bound, that the Weierstrass condition has to be referred to the line  $m'$  of angular coefficient  $-\mu_s^*$  [Fig. 13(b)] instead of the slope  $m$  at  $q_\ell = q_{\ell \min}$ . Then again, as indicated in the drawing, the W-test is satisfied for any admissible  $\Delta q_\ell > 0$  with respect to  $q_\ell = q_{\ell \min}$ . Note that in Fig. 13(b) at most the lines  $m'$  and  $n'$  may coincide with the slopes  $m$  and  $n$ . For a control  $q_\ell = q_{\ell Q}$ , since the admissible variations are  $\delta q_\ell \leq 0$  then Eq. (96) applies and the W-test must be referred to the slope  $b$  [Fig. 13(b)].

All these considerations may be graphically summarized in what is called "Cicala's pointer" construction. Such construction is indicated in Fig. 13(c). The angle between the pointer and the horizontal correspond to values  $-(\mu_2 \varphi_2 / \mu_1)$  along the extremal. The directions  $m$  and  $n$  correspond to the slopes of the characteristic line at the boundaries  $q_\ell = q_{\ell \max}$ ,  $q_\ell = q_{\ell \min}$ . When the pointer is in the interior of the shadowed circular sector in Fig. 13(c), (e.g., position  $b$  drawn) it indicates that the control is  $q_{\ell \min} < q_\ell < q_{\ell \max}$ . This circular sector may change its angle in time, of course, if the characteristic line  $\varphi_1(r, q_j, q_\ell)$  changes its form along the extremal and/or the boundaries are time-dependent. For values  $-(\mu_2 \varphi_2 / \mu_1)$  corresponding to directions of the pointer as indicated with  $m'$  or  $n'$  in Fig. 13(c), the control operation is at the boundaries as shown in the picture. The case of control at the boundary  $q_{\ell \max}$  using the H-line representation is shown in Fig. 13(a). Since for  $q_\ell = q_{\ell \max}$  only variations  $\delta q_\ell \leq 0$  are admissible then Eq. (88c) applies. This is shown at point R in Fig. 13(a) where  $H_{q_\ell} > 0$ . At point R in Fig. 13(b) assume the Weierstrass condition is satisfied with  $\mu^* > -\frac{\partial \varphi_1}{\partial q_\ell}$  and for any admissible one-sided increment  $\Delta q_\ell < 0$ . The condition  $W$  at R [Fig. 13(b) line  $n'$ ] that  $W > 0$ , corresponds with the condition  $H_{q_\ell} > 0$  at R in Fig. 13(a). At most  $n'$  may coincide with  $n$  [Fig. 13(b)], and then in such case we would have  $H_{q_\ell} = 0$  for  $q_\ell = q_{\ell \max}$ .

This limiting situation corresponds with the condition at point R' in Fig. 13(a). The sequence of H-lines in Fig. 13(a) has been drawn to present an example of how it would look a "smooth transition" or

"junction of sub-arcs without corners", considering time-dependent  $H$ -lines at different times along the extremal. The time  $\tau = \tau_c$  indicates the time at the junction. At the time  $\tau_c - \Delta\tau$  assume the control is  $q_{\min.} < q_{R''} < q_{\max.}$ , and then the corresponding  $H$ -line will look as shown in Fig. 13(a). At the time  $\tau = \tau_c$  the control is  $q_{R'} = q_{\max.}$  and at  $\tau = \tau_c + \Delta\tau$  the control continues on the boundary  $q_{\max.}$ . Since at  $\tau = \tau_c$ ,  $q(\tau_c - 0) = q(\tau_c + 0)$ , then the slopes  $q'_j(\tau)$  are continuous at the corner and the transition occurs without a vertex on the extremal. Evidently, here it has been assumed a time varying  $H$ -line and fixed boundaries. Points  $P$  and  $M$  on the  $H$ -line at time  $\tau_c + \Delta\tau$  represent operating conditions satisfying  $H_{q\ell} = 0$  but discardables because they do not satisfy  $\Delta H \leq 0$  for all admissible  $\Delta q_\ell$ .

### 3. Graphical Interpretation of the Corner Point Conditions.

In this paragraph the case of a transition between extremal sub-arcs with a control jump  $\Delta q_\ell$ , will be considered. Since the necessary conditions at corners of the extremal arc have been analyzed in previous developments, here it will only be made a summary of such requirements to the extent of their graphical interpretation. From Eq. (24) is seen that at corners of the extremal arc and for the specific type of constraints being considered, [Eq. (80)], the canonical variables  $v_j$  and multipliers  $\mu_j$  must be continuous. Thus, the index-value  $\mu^* = \frac{\mu_2 \varphi_2(q_j, \tau)}{\mu_1}$  must be continuous. Eq. (25) indicates that the Weierstrass condition vanish at corners. This was previously considered in Section 4. Consequently, from Eqs. (24), (25) and previous considerations we may conclude that a corner point, with a control jump  $\Delta q_\ell$ , may exist if

- a. The characteristic line has two points on it with a common tangent, i.e., a bitangent. This case is pictured in Fig. 14. Note that the controls  $q_R$  and  $q_P$  correspond to the points of contact between the bitangent  $b$  and the characteristic line. The controls  $q_R$  and  $q_P$  may be interior to the interval of admissible control or either one of them (or even both) may be on the boundary. Points  $P$  and  $R$  have the same index-value  $\mu^*$  and the

Weierstrass condition with respect to either one of them vanish at the other, for the increment  $\Delta q_\ell = q_{\ell R} - q_{\ell P}$  or  $\Delta q_\ell = q_{\ell P} - q_{\ell R}$  according to the point taken.

- b. The tangent to the characteristic line at a point whose control is interior to the interval  $(q_{\ell \max.}, q_{\ell \min.})$  intersects the characteristic line again at a point whose control is on the boundary. This case is shown in Fig. 15(b) and was also presented in Fig. 7(b). Note that in Fig. 15(b), point P satisfies the Weierstrass condition with  $\mu_P^* = -\partial \varphi_1 / \partial q_\ell \big|_P$  since at P,  $\delta q_\ell \geq 0$ . At point R the Weierstrass condition is satisfied with  $\mu_R^* < -\partial \varphi_1 / \partial q_\ell \big|_R$ , viz., it is referred to the line  $m'$  since the admissible control variation is now  $\delta q_\ell \geq 0$ . Again, the Weierstrass condition vanishes at P or R when taken with respect to R or P respectively. The continuity of the index-value, (viz.,  $\mu_R^* = \mu_P^*$ ), is satisfied.

The corresponding situation may be represented applying the H-lines. From Eq. (52) we find that at a corner C of the extremal arc,  $H(r_c - 0) - H(r_c + 0) = 0$ , which is also found from Eq. (77). Consequently, at P and R it must be  $\Delta H = 0$ , that is  $H_P = H_R$ . The case is shown in Fig. 15(a). Since  $q_{\ell P}$  is interior to the interval  $(q_{\ell \max.}, q_{\ell \min.})$  then  $\delta q_\ell \geq 0$  are admissible and thus  $H_{q_\ell} \big|_P = 0$  holds. However at R, since only  $\delta q_\ell \geq 0$  are admissible and  $m'$  is not coincident with  $m$  in Fig. 15(b), then  $H_{q_\ell} \big|_R < 0$  holds. Note that also the condition  $H_R = H_P$  is satisfied.

Finally, since it has been assumed that the extremal arc has a corner C at  $r = r_c$ , the broken extremal  $q_i(r)$  is shown in Fig. 15(c). Typical configurations of the characteristic line and of the H-lines at different times  $r$  along the extremal E are shown in Figs. 15(a), (b) and (c). Since the drawings are felt to be self-explanatory other considerations are left to the reader.

To conclude this graphical interpretation of the necessary conditions along the extremal arc we should mention the case most frequently found in problems of the mechanics of flight of airplanes, gliders, and rocket vehicles, i.e., the case where  $\frac{\partial H}{\partial r} = 0$ . In such case and from Eq. (52)

$$\left[ H(r, q_j(r), q_\ell(r), v_j(r)) \right] = \text{const.} \quad (107)$$

along the extremal. Therefore, assuming a sub-arc along which the control  $q_\ell(r)$  satisfies  $q_{\ell \min}(r) < q_\ell(r) < q_{\ell \max}(r)$  [see Fig. 16(b)] the condition (107) will be satisfied along the extremal as indicated in Fig. 16(a) at the points P, Q, R on the corresponding H-lines at times  $\tau_1, \tau_2, \tau_3$ . The previous points satisfy the conditions  $H_P = H_Q = H_R$ , and  $H_{q_\ell} = 0$ ,  $q_{\ell \min} < q_\ell < q_{\ell \max}$ , at each one of them. For cases of control on the boundary of course  $H_{q_\ell}$  must satisfy Eqs. 88(b) or (c). In Fig. 16(a) it has been assumed that at points P, Q, and R the Weierstrass condition is satisfied in its strengthened form.

## F. DISCONTINUOUS SOLUTIONS - APPLICATIONS

In this section some applications of the previous developments will be considered in connection with some interesting problems of the mechanics of flight. The following problems constitute interesting examples of discontinuous extremal solutions.

### 1. Example I. Optimum Glide Trajectories in an Uniform Gravitational Field.

This problem has been considered by this author in Ref. 21. In the light of the hypotheses introduced in Ref. 21, the equations of constraint are written (glide trajectory with shallow angle of descent and centripetal acceleration negligible)

$$\phi_1 \equiv \xi' - Z\varphi(a) = 0 \quad (108)$$

$$\phi_2 \equiv [Z' + \mathcal{D}(Z, \tilde{h})] Z + \tilde{h}' = 0 \quad (109)$$

$$\phi_3 \equiv \tilde{h}' - Z a = 0 \quad (110)$$

The problem is to determine the optimum solution  $\xi(r)$ ,  $Z(r)$ ,  $\tilde{h}(r)$ ,  $a(r)$  minimizing  $\Lambda = \Lambda(\xi_F, Z_F, \tilde{h}_F, r_F, \xi_I, Z_I, \tilde{h}_I, r_I)$  and satisfying a given set of end-conditions (Ref. 21). The nomenclature is explained in the list of symbols. The control variable  $a(r)$  is bounded,  $-1 \leq a \leq +1$ , and  $\varphi(a) \geq 0$ , so as to assure a monotonically increasing non-dimensional range  $\xi(r)$ . The characteristic line is obtained from the Weierstrass condition

$$W \equiv -Z\mu_1 [\varphi - \varphi^* + \mu^*(a - a^*)] \geq 0 \quad (111)$$

where  $\mu^*$  is the index-value  $\mu^* = \frac{\mu_3}{\mu_1}$ , and the Euler equations

$$\mu_1' = 0, \quad \mu_1 \frac{d\varphi}{da} + \mu_3 = 0 \quad (112)$$

The characteristic line, defined by the function  $\varphi(a) = (1 - a^2)^{1/2}$  is shown in Fig. 17. It may be shown that the Legendre condition leads to the requirement

$$-\mu_1 \frac{d^2 \varphi}{da^2} \geq 0 \quad (113)$$

Since  $d^2 \varphi / da^2 = -(\varphi^2 + a^2) / \varphi^3 < 0$ , then Eq. (113) requires that  $\mu_1 \geq 0$ . Note that the characteristic line in this case is not time-dependent; thus, it may be drawn once for all and be used at any given time  $\tau$ .

Now assume that point P in Fig. 17 represents the point on the characteristic line corresponding to the conditions at  $(\tau, q_j)$  on an extremal arc. Since we assume  $(\tau, q_j) \equiv (\tau, \xi, Z, h)$  given, then from Eqs. (108) to (110) the slopes  $(q'_j) \equiv (\xi', Z', h')$  may be determined at P for the control value  $a = a_P$ . Also, the slope  $(d\varphi/da)_P$  is determining the index-value  $\mu^*_P$ . The Legendre condition is assumed satisfied in its strengthened form  $\mu_1 > 0$ .

If the Weierstrass condition Eq. (111) is applied at point P, for arbitrary control increments  $\Delta a$  it may be seen that  $W > 0$  in any case (e.g., see at points P' and P'' in Fig. 17). However, if point R on the characteristic line were assumed as representing the conditions at  $(\tau, q_j)$  on the extremal, it may be verified that the Weierstrass condition is also satisfied for any admissible  $\Delta a$  assumed. Of course, now the slopes and multipliers  $(q'_j, \mu_j)_R$  differ from the set  $(q'_j, \mu_j)_P$ . A similar situation will be encountered at any other point on the characteristic line. Thus, at  $(\tau, q_j) \equiv (\tau, \xi, Z, h)$  on the extremal, there are sets  $(q'_j, \mu_j) \equiv (\xi', Z', h', \mu_1, \mu_2, \mu_3)$  with which the Weierstrass condition is satisfied for control values  $q_\ell = a$ , and for any admissible control increment  $\Delta q_\ell = \Delta a$ .

The Hamiltonian function H, in our case is written

$$H(\tau, q_j, a, v_j) = v_1 Z \varphi(a) - v_2 \left[ a + \mathcal{P}(Z, h) \right] + v_3 Z a \quad (114)$$

where the canonical variables  $v_1(\tau), v_2(\tau), v_3(\tau)$  are related to the multipliers by the transformation

$$\begin{aligned}
v_1 &= \mu_1 \\
v_2 &= \mu_2 Z \\
v_3 &= \mu_2 + \mu_3
\end{aligned} \tag{115}$$

From Eqs. (48), (49) and (114) is obtained

$$v_1' = -\frac{\partial H}{\partial \xi} = 0 \tag{116}$$

$$v_2' = -\frac{\partial H}{\partial Z} = -v_1 \varphi(a) + v_2 \mathcal{D}_Z - v_3 a \tag{117}$$

$$v_3' = -\frac{\partial H}{\partial h} = v_2 \mathcal{D}_h \tag{118}$$

$$\frac{\partial H}{\partial a} = v_1 Z \frac{d\varphi}{da} - v_2 + v_3 Z = 0, \text{ (for } \delta a \begin{smallmatrix} < \\ > \end{smallmatrix} 0 \text{ admissible).} \tag{119}$$

which are the canonical differential equations of the extremals. From Eqs. (116) to (119) and relations (115) is obtained.

$$\mu_1' = 0 \tag{120}$$

$$\mu_2' Z = -\mu_1 \varphi(a) + \mu_2 (\mathcal{D} + Z \mathcal{D}_Z) - \mu_3 a \tag{121}$$

$$\mu_2' + \mu_3' = \mu_2 Z \mathcal{D}_h \tag{122}$$

$$\mu_1 \frac{d\varphi}{da} + \mu_3 = 0, \text{ (for } \delta a \begin{smallmatrix} < \\ > \end{smallmatrix} 0 \text{ admissible)} \tag{123}$$

which are the Euler differential equations of the set of constraints Eqs. (108) to (110):

The Hamiltonian  $H$  may be rewritten as

$$H(r, q_j, a, v_j) = v_1 Z \left[ \varphi(a) + v^* a - v_2 \mathcal{D}(Z, \tilde{h}) \right] \quad (124)$$

In Eq. (124) the term  $v^* = \frac{v_3 Z - v_2}{v_1 Z}$  is the canonical expression of the index-value. As a matter of fact, from Eqs. (115) is obtained

$$v^* = \frac{v_3 Z - v_2}{v_1 Z} = \frac{(\mu_2 + \mu_3) Z - \mu_2 Z}{\mu_1 Z} = \frac{\mu_3}{\mu_1} = \mu^*$$

Now, the  $\Pi$ -function for the set of constraints [Eqs. (108) to (110)] considered is written

$$\begin{aligned} \Pi(q'_j, q_j, \mu_i, r) = & \mu_1 \left[ \xi' - Z \varphi(a) \right] + \mu_2 \left\{ \left[ Z' + \mathcal{D}(Z, \tilde{h}) \right] Z + \right. \\ & \left. + \tilde{h}' \right\} + \mu_3 (\tilde{h}' - Z a) \end{aligned}$$

Then, from Eq. (19), since  $\Pi_r = 0$ , it follows

$$\Pi - q'_j \Pi_{q'_j} = M = \text{const.}$$

and therefore

$$\mu_1 \varphi(a) - \mu_2 \mathcal{D}(Z, \tilde{h}) + \mu_3 a = - \frac{M}{Z} \quad (125)$$

Consider now the minimum time problem (brachistochronic problem) with unspecified range. From the Transversality Condition, Eq. (11), and assuming a normal extremal ( $\lambda_0 = 1$ ), it follows

$$\begin{aligned} d\Lambda \left[ r_I, r_F, q_j(r_I), q_j(r_F) \right] + \left[ \mu_1 d\xi + \mu_2 Z dZ + (\mu_2 + \mu_3) d\tilde{h} + \right. \\ \left. + M dr \right]_I^F = 0 \end{aligned}$$

which is to be satisfied for any admissible set of variations

$(dq_{j,I,F}, dr_{I,F}) \neq (0, 0)$  consistent with the equations of variation on  $E$ ,

$$\Psi_{\rho}(dq_I, dq_F, dr_I, dr_F) = 0$$

Assume that the following boundary conditions  $\left[ \psi_{\rho} = 0, \quad \rho = 1, \dots, r \leq 7 \right]$  are imposed

$$\begin{aligned} \psi_1 &\equiv r_I = 0 & \psi_4 &\equiv \bar{h}_I - b_I = 0 \\ \psi_2 &\equiv Z_I - a_I = 0 & \psi_5 &\equiv \bar{h}_F - b_F = 0 & a_I, a_F, b_I, b_F &= \text{const.} \\ \psi_3 &\equiv Z_F - a_F = 0 & \psi_6 &\equiv \xi_I = 0 \end{aligned}$$

Since the minimum time problem is being considered, the functional to be minimized is written

$$\Lambda \left[ r_I, r_F, q_{j_I}, q_{j_F} \right] \equiv r_F - r_I$$

Therefore from the Transversality Condition and the boundary conditions assumed the following sub-conditions may be derived

$$(1 + M) dr_F = 0, \quad dr_F \neq 0 \quad \text{then } M = -1 = \text{const.}$$

$$\mu_1 d\xi_F = 0, \quad d\xi_F \neq 0 \quad \text{then } \mu_1 = K_1 = 0 = \text{const.}$$

Consequently, since  $\mu_1 = K_1 = 0$ , from Eq. (123) is obtained  $\mu_3 = 0 = \text{const.}$  As may be seen, the index-value  $\mu^* = \mu_3/\mu_1$  for this problem is undetermined. However, the optimum control operation may be determined from the Euler equations, as will be shown in the following.

The first integral, Eq. (125), leads now to  $(\mu_1 = \mu_3 = 0)$

$$\mu_2 = -\frac{1}{Z\mathcal{D}(Z, \bar{h})}$$

From the previous equation and Eqs. (108) to (110) the following derivative may be obtained

$$\mu_2' = -\frac{1}{\mathcal{D}^2 Z^2} \left[ \beta a + \mathcal{D}(\mathcal{D} + Z\mathcal{D}_Z) \right], \quad \beta = \mathcal{D} + Z\mathcal{D}_Z - Z^2\mathcal{D}_{\bar{h}}$$

Also, from Eq. (121) and for  $\mu_1 = \mu_3 = 0$ ,

$$\mu_2' = \frac{\mu_2}{Z} (\mathcal{D} + Z\mathcal{D}_Z)$$

Consequently, equating the right hand side members of previous equations is derived

$$\mu_2 (\mathcal{D} + Z\mathcal{D}_Z) + \frac{1}{\mathcal{D}^2 Z} \left[ \beta a + \mathcal{D} (\mathcal{D} + Z\mathcal{D}_Z) \right] = 0$$

which, accounting for  $\mu_2 = -1/Z \mathcal{D}$  leads to

$$\mu_2 (\mathcal{D} + Z\mathcal{D}_Z) - \mu_2 \left[ \frac{\beta a}{\mathcal{D}} + (\mathcal{D} + Z\mathcal{D}_Z) \right] = 0$$

and thus

$$\mu_2 \frac{\beta a}{\mathcal{D}} = 0$$

Since the aerodynamic drag  $\mathcal{D} \neq 0$  and  $\mu_2 = \frac{1}{Z\mathcal{D}} \neq 0$ , the preceding equation is indicating that the extremal arc may be discontinuous and composed of sub-arcs

$$a) \quad \beta(Z, \tilde{h}) = \mathcal{D} + Z\mathcal{D}_Z - Z^2 \mathcal{D}_{\tilde{h}} = 0, \quad a \neq 0$$

$$b) \quad a = 0, \quad \beta \neq 0$$

Since the discontinuous extremal solution has been obtained in closed-form expression the sub-arcs forming part of the extremal arc may be determined after drawing the sub-arc  $\beta(Z, \tilde{h}) = 0$  in the  $(Z, h)$ -plane and locating the boundary values (boundary value problem)

$$I(Z_I, \tilde{h}_I) = I(a_I, b_I), \quad Z_I > Z_F,$$

$$F(Z_F, \tilde{h}_F) = F(a_F, b_F), \quad \tilde{h}_I > \tilde{h}_F,$$

The sub-arcs  $a=0$  correspond with  $\tilde{h} = \text{const.}$  sub-arcs, [from Eq. (110)], along which  $Z' < 0$ , [from Eq. (109)]. Typical cases of discontinuous solutions, for the problem discussed, are shown schematically in Figure 18. A more detailed analysis of this type of problem has been offered by this author in Ref. 21. The arrows in Fig. 18 indicate the sense of displacement along the extremal.

Fig. 18 shows the solution to different boundary value problems. Note that according to the boundary values assigned the number of sub-arcs forming the discontinuous extremal may change and the solution may be continuous or discontinuous.

## 2. Example II. Optimum Trajectories in a Central Gravitational Field.

This example will present a so-called "degenerate" case, where the characteristic lines are linear. A detailed analysis of this case has been undertaken by this author in Ref. 18, where sufficiency conditions based on the Jacobi Theorem on Conjugate points have been developed.

In the light of the hypotheses made in Ref. 18, (equilibrium glide trajectories about a planet), the equations of motion are written

$$\phi_1 = \xi' - Z = 0 \quad (126)$$

$$\phi_2 = \tilde{h}' - Z\theta = 0 \quad (127)$$

$$\phi_3 = Z' + \mathcal{D}(Z, \tilde{h}, \mathcal{L}) = 0 \quad (128)$$

$$\phi_4 = \mathcal{L} - (1 - Z^2) = 0 \quad (129)$$

The present problem is that of finding the optimum solution  $\xi(r), \tilde{h}(r), Z(r), \mathcal{L}(r), \theta(r)$  minimizing a functional  $\Lambda = \Lambda(\xi_F, \tilde{h}_F, Z_F, r_F, \xi_I, \tilde{h}_I, Z_I, r_I)$  and satisfying a set of given end conditions  $\psi_\rho(q_{j_F}, q_{j_I}, r_F, r_I) = 0, \rho = 1, \dots, r \leq 7$ , (see Ref. 18).

From Eq. (39) is readily seen that the Legendre condition, for the set of constraints Eqs. (126) to (129), is written

$$\mu_3 \mathcal{D}_{\mathcal{L}\mathcal{L}} \geq 0 \quad (130)$$

Since for a quadratic drag function  $\mathcal{D} = \mathcal{D}_0(Z, \tilde{h}) + \mathcal{D}_1(Z, \tilde{h})\mathcal{L}^2, \mathcal{D}_{\mathcal{L}\mathcal{L}} = 2\mathcal{D}_1(Z, \tilde{h}) > 0$ , then condition (130) requires

$$\mu_3(r) \geq 0 \quad (131)$$

along the extremals. The Weierstrass condition is written

$$W = -\mu_2 Z (\theta - \theta^*) \geq 0 \quad (132)$$

Now the characteristic line is defined by  $\varphi(\theta) = \mu_2 Z \theta$ , and is drawn in Fig. 19(a) for different values of  $\mu_2$ . Each characteristic line corresponds to a value of the multiplier  $\mu_2$ . However, for each given value  $\mu_2 \neq 0$ , and for unrestricted admissible variations  $\delta\theta \gtrless 0$ , it may be seen that the Weierstrass condition is not satisfied [see points P' and P'' in Fig. (19a)]. For example, for  $\mu_2 > 0$  [see Fig. 19(a)] at point P' the W-test is not satisfied for control increments  $\Delta\theta > 0$ .

The Hamiltonian is written (notice that now the canonical variable  $v_4 = \Pi_Q$  is introduced)

$$H(r, q_j, \theta, v_j) = v_2 Z \theta + v_1 Z - v_3 \mathcal{D}(Z, \tilde{h}, \mathcal{L}) - v_4 \phi_4 \quad (133)$$

where the canonical variables  $v_j$  are related to the multipliers by

$$\begin{aligned} v_1 &= \mu_1 & v_3 &= \mu_3 \\ v_2 &= \mu_2 & v_4 &= \mu_4 \end{aligned} \quad (134)$$

Thus, the H-lines, corresponding to  $v_2 \gtrless 0$  (i.e.,  $\mu_2 \gtrless 0$ ) may be drawn at given  $(r, q_j)$  as shown in Fig. 19(b). From Figs. 19(a) and (b) it may be readily seen that the Weierstrass condition ( $W \geq 0$ ) and the Maximality Principle  $\Delta H \leq 0$ , for unrestricted admissible variations  $\delta\theta \gtrless 0$ , may only be satisfied in the form  $W=0, \Delta H=0$  for  $\mu_2=0$  and  $v_2=0$  respectively. The optimum control operation  $\theta^*(r)$  apparently is undetermined.

However, for the problem in consideration the control value  $\theta^*$  may be derived from the differential equations of the extremals as will be shown. In fact from Eqs. (48), (49), (129) and (133) is obtained

$$v_1' = - \frac{\partial H}{\partial \xi} = 0 \quad (135)$$

$$v_2' = - \frac{\partial H}{\partial \tilde{h}} = v_3 \mathcal{D}_{\tilde{h}} \quad (136)$$

$$v_3' = - \frac{\partial H}{\partial Z} = - v_2 \theta - v_1 + v_3 \mathcal{D}_Z + 2v_4 Z \quad (137)$$

$$\frac{\partial H}{\partial \mathcal{L}} = - v_3 \mathcal{D}_{\mathcal{L}} - v_4 = 0 \quad (138)$$

and since  $\delta\theta \geq 0$ , are admissible, then

$$\frac{\partial H}{\partial \theta} = v_2 Z = 0 \quad (139)$$

The preceding equations are the canonical differential equations of the extremals. Using relations (134), and from the previous equations is immediately obtained

$$\mu_1 = K_1 = \text{const.} \quad (140)$$

$$\mu_2' = \mu_3 \mathcal{D}_{\tilde{h}} \quad (141)$$

$$\mu_3' = - \mu_2 \theta - \mu_1 + \mu_3 \mathcal{D}_Z + 2\mu_4 Z \quad (142)$$

$$0 = \mu_3 \mathcal{D}_{\mathcal{L}} + \mu_4 \quad (143)$$

$$0 = \mu_2 Z \quad (144)$$

which are the Euler equations of the set of constraints given by Eqs. (126) to (129), (see Ref. 18). Finally, for  $\mu_2(r) = 0$  from the condition (131) and Eq. (141) is obtained

$$\mathcal{D}_{\tilde{h}}(Z, \tilde{h}, \mathcal{L}) = 0. \quad (145)$$

As shown in Ref. 18, from the expression of the non-dimensional drag  $\mathcal{D}(Z, \tilde{h}, \mathcal{L})$ , Eq. (145) and Eq. (129) the extremal sub-arc (145) may be explicitly written

$$f(Z, \tilde{h}) \equiv Z^2 \left[ k\eta \left( \frac{C_{D_o} \frac{d\eta}{d\tilde{h}} + \eta \frac{dC_{D_o}}{dM} \frac{\partial M}{\partial \tilde{h}}}{K \frac{d\eta}{d\tilde{h}} - \eta \frac{dK}{dM} \frac{\partial M}{\partial \tilde{h}}} \right)^{1/2} + 1 \right] - 1 = 0 \quad (146)$$

Therefore, from Eqs. (126) to (129) and (146) the optimum control may be obtained as

$$\theta = \mathcal{D} \left[ Z, \tilde{h}, \mathcal{L}(Z) \right] \frac{f_Z}{Z f_{\tilde{h}}} = \theta(Z, \tilde{h}) \quad (147)$$

Then for any given admissible set  $(r, \xi, Z, \tilde{h})$  on the extremal, the control value  $\theta^*$  [Eq. (147)] may be directly determined, thus avoiding the problem of integrating the multipliers or canonical variables  $v_j$ , since they have been eliminated. It is to be noted that according to the assumptions in Ref. 18 it is required that  $|\theta| \leq \theta_0 \ll 1$ , (of course  $\theta_0$  is not prescribed). This is to be verified in each application "a posteriori", to assert the compliance of the extremal with the physical assumptions. In Ref. 18, in effect is shown that along the extremal the angle  $\theta$  is very small and thus consistent with the hypotheses made.

To conclude, it may be shown that the extremal arc may be composed of sub-arcs  $\theta=0$  and sub-arcs  $\theta = \text{variable}$ . The combination or number of sub-arcs forming the extremal arc depends on the boundary conditions imposed. A more detailed analysis of these aspects is offered in Ref. 18.

The admissibility of the discontinuous solution may be readily detected. In fact, from Eq. (52) and Eq. (133) is derived ( $v_2 = 0$ )

$$\begin{aligned} \frac{dH}{dr} &= v_1' Z + v_1 Z' - v_3' \mathcal{D} - v_3 (\mathcal{D}_Z Z' + \mathcal{D}_{\tilde{h}} \tilde{h}' + \mathcal{D}_{\tilde{p}} \tilde{p}') \\ &= (v_2 \mathcal{D} - v_3 \mathcal{D}_{\tilde{h}} Z) \theta = 0 \end{aligned}$$

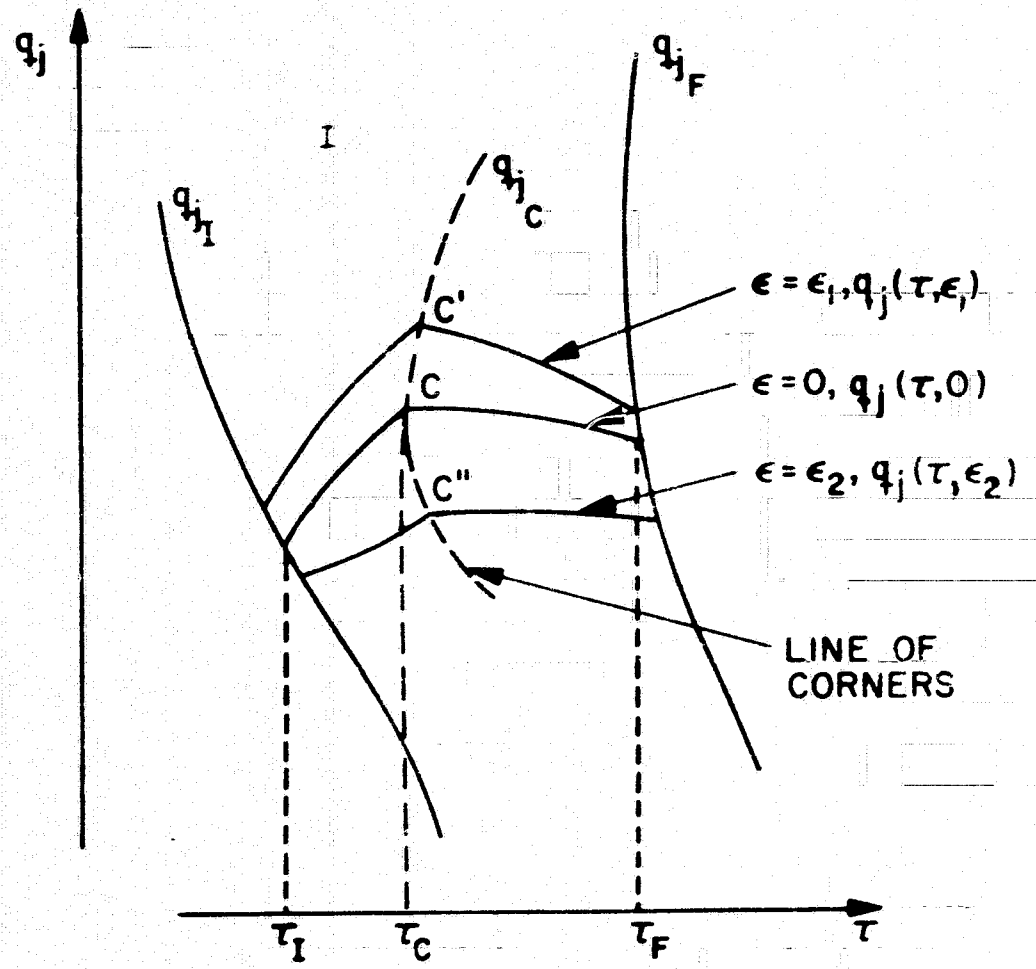
and therefore

$$v_3 \mathcal{D}_{\tilde{h}} Z \theta = 0, \quad (v_3 > 0, Z > 0) \quad (148)$$

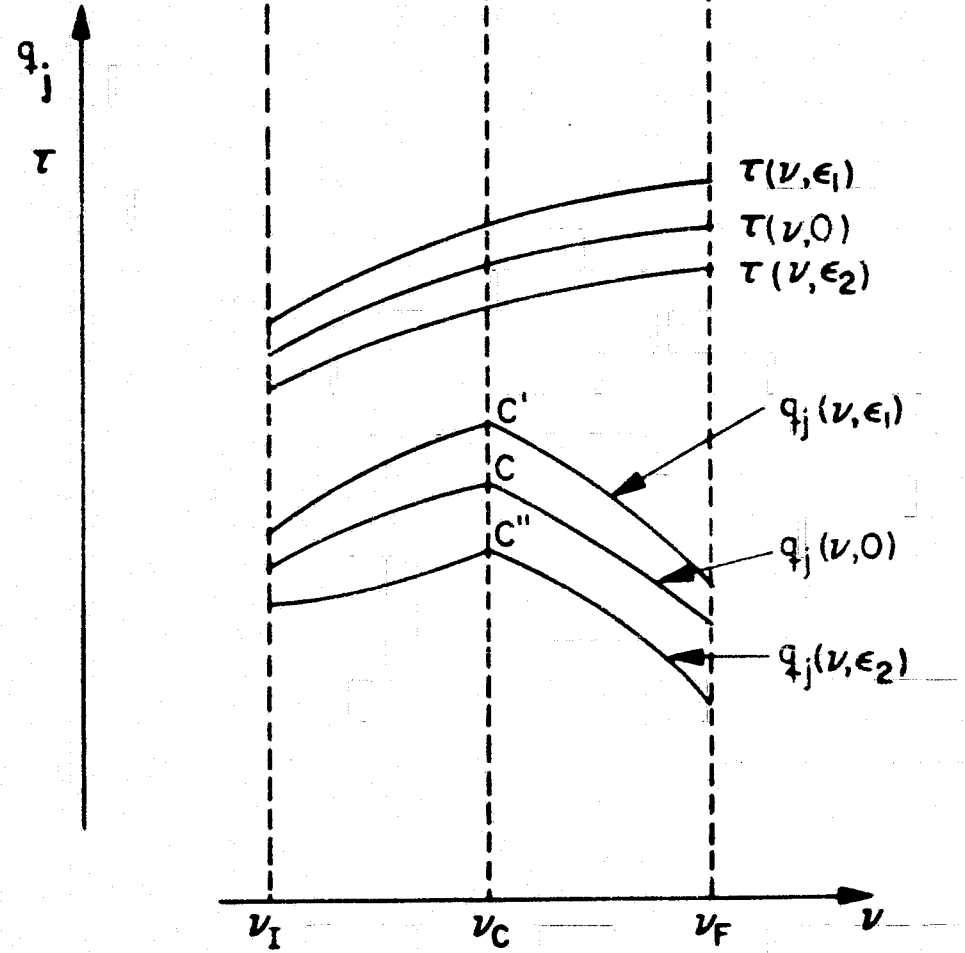
Eq. (148) shows that the extremal arc may be discontinuous and composed of sub-arcs

$$\begin{aligned} \text{a)} \quad & \mathcal{D}_{\tilde{h}} [Z, \tilde{h}, \tilde{p}(Z)] = 0, \quad \theta = \text{variable.} \\ \text{b)} \quad & \theta = 0 = \text{const.}, \quad \mathcal{D}_{\tilde{h}} \neq 0 \end{aligned} \quad (149)$$

The discontinuous extremal solution (149) physically means that the optimum transfer may be composed of circular lifting orbits ( $\theta = 0$ ), sub-arcs and lifting transfer ( $\mathcal{D}_{\tilde{h}} = 0$ ) sub-arc. The number of sub-arcs forming the extremal depends on the boundary conditions imposed (boundary value problem) in the  $(Z - \tilde{h})$ -plane. Several typical boundary value problems involving different optimum transfers are shown in Fig. 20.



(a)



(b)

FIGURE 1. ONE-PARAMETER FAMILY OF ADMISSIBLE ARCS FOR THE CASE OF PARAMETRIC AND NON-PARAMETRIC VARIATIONAL FORMULATIONS.



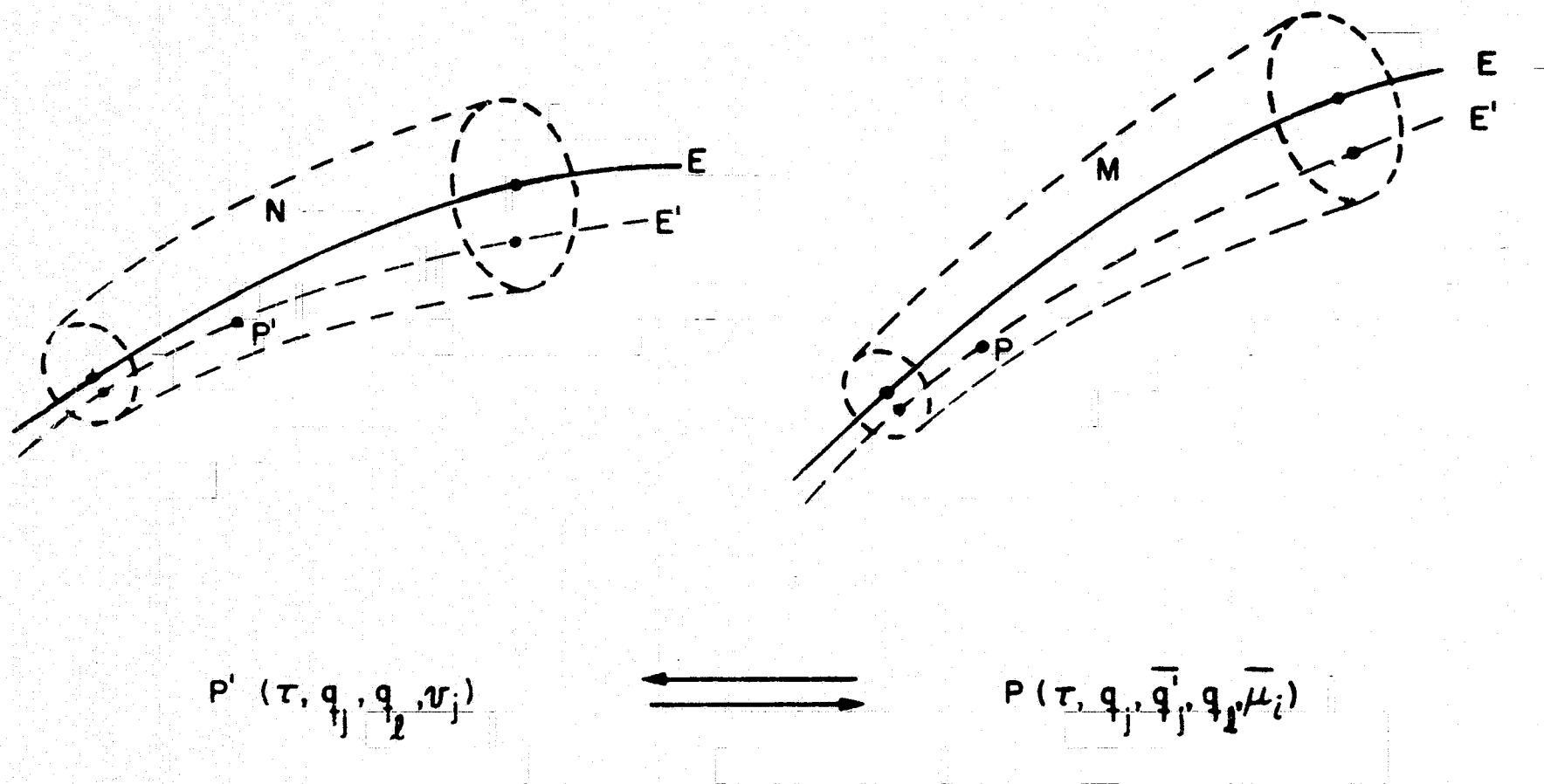


FIGURE 3. ONE-TO-ONE CORRESPONDENCE BETWEEN THE POINT  $P (\tau, q_j, \bar{q}_j, q_{\ell}, \bar{\mu}_i)$  IN M AND THE POINT  $P' (\tau, q_j, q_{\ell}, v_j)$  IN N.

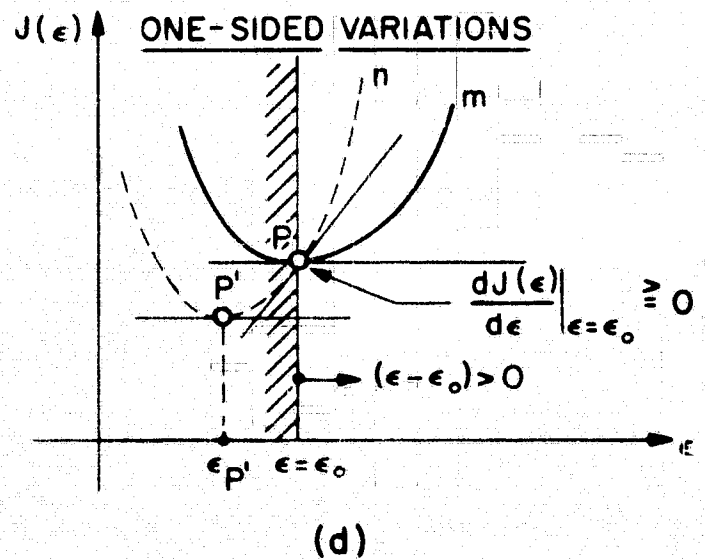
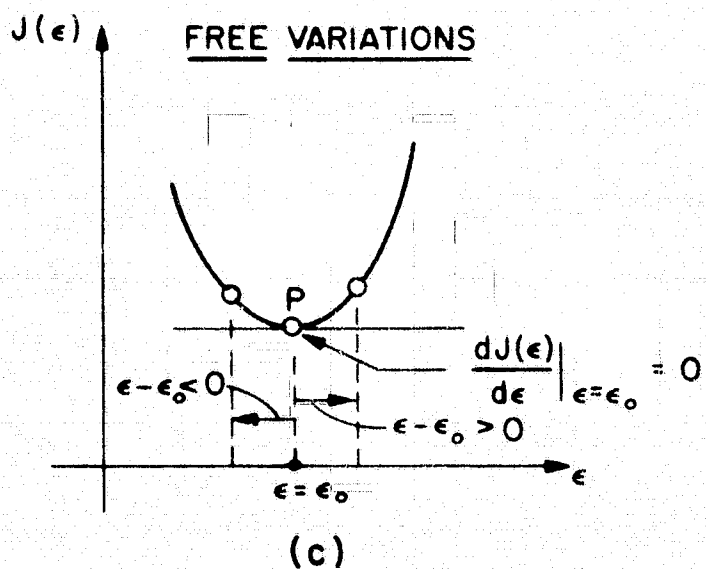
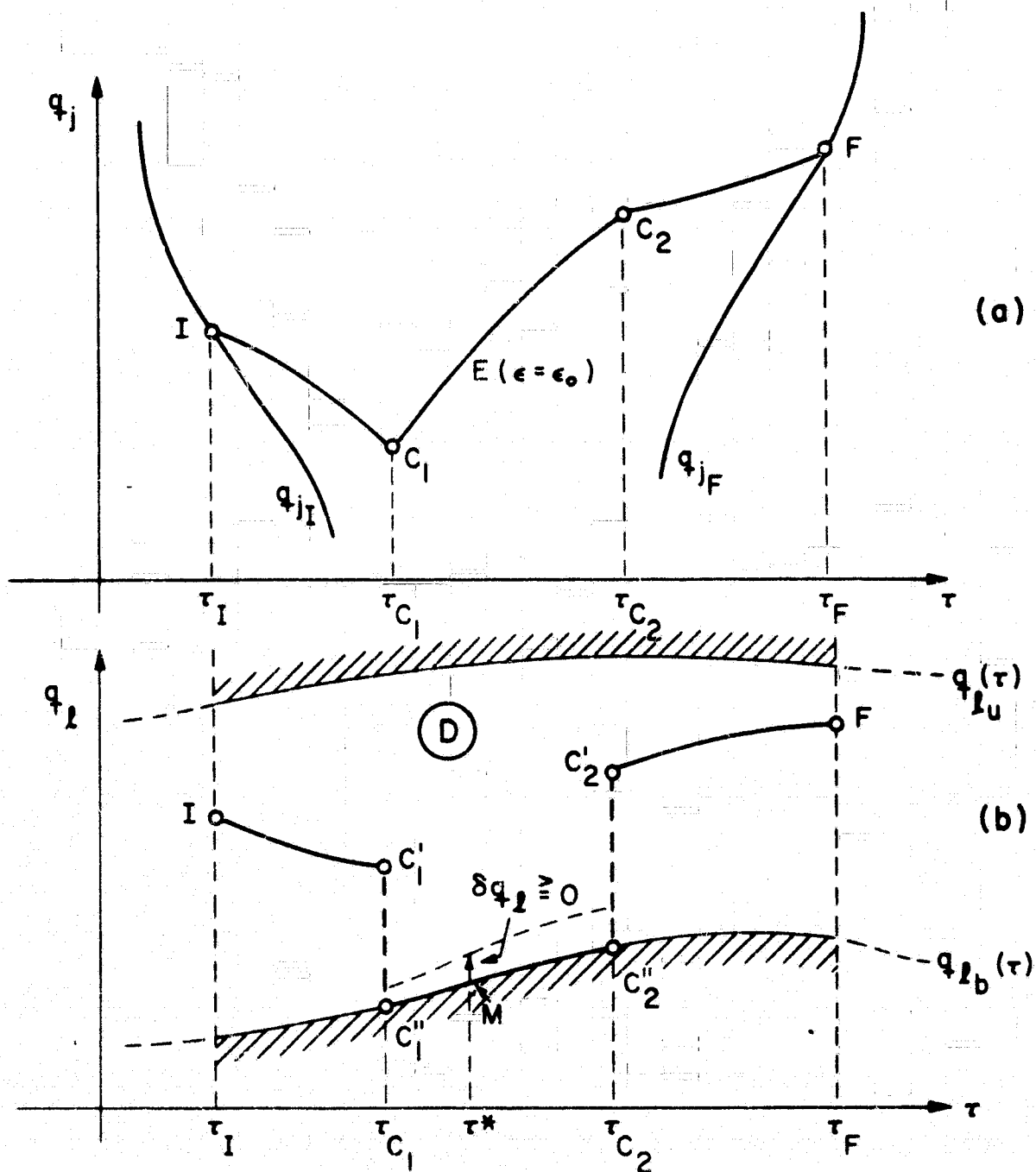
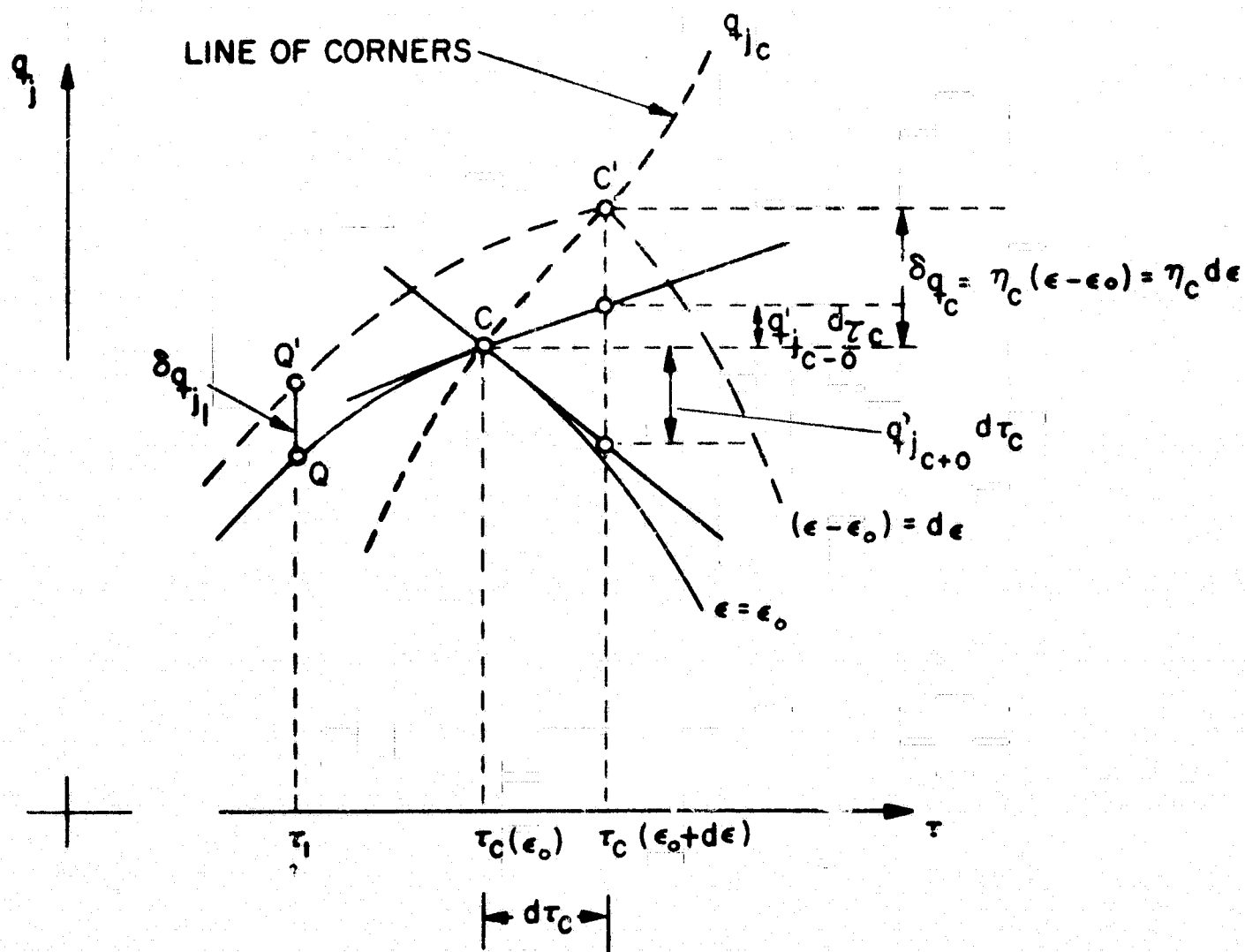


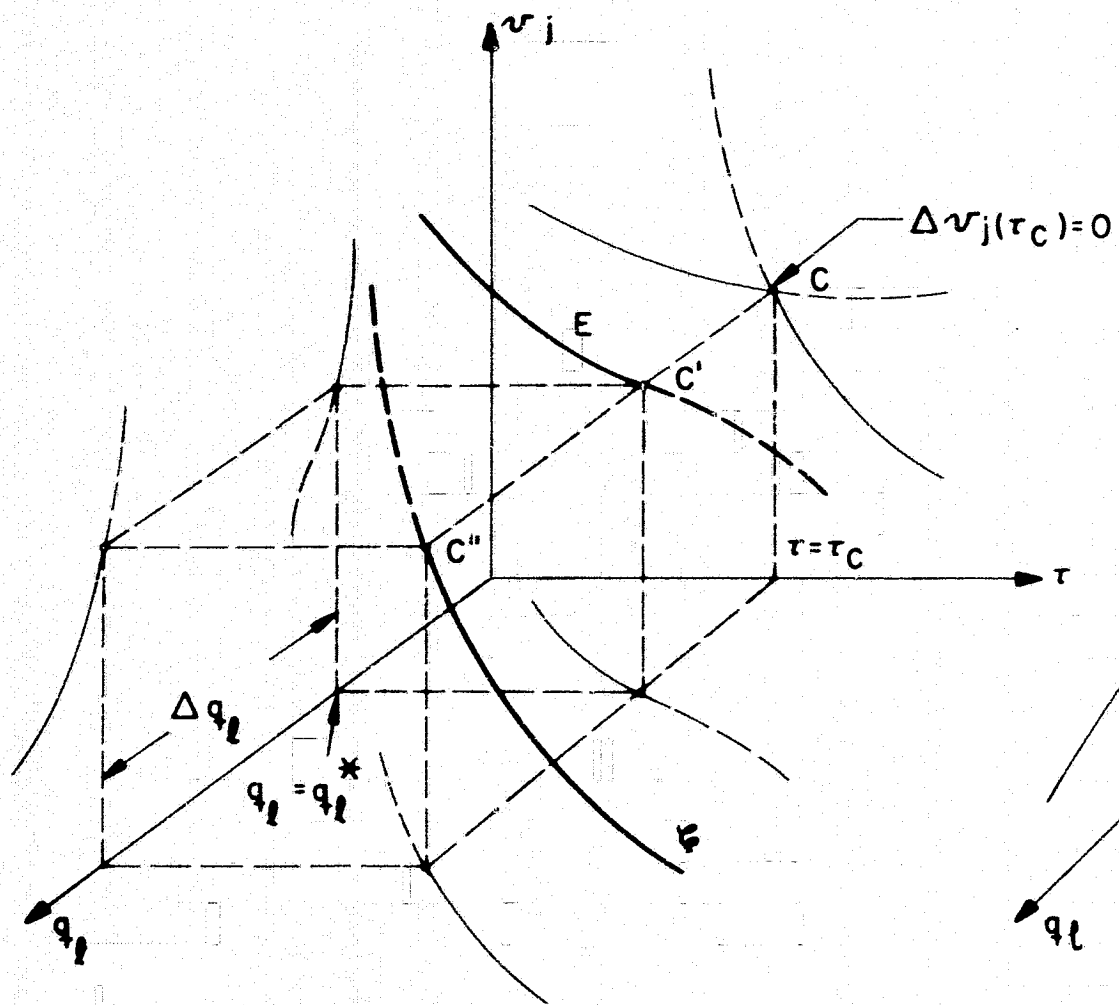
FIGURE 4. BROKEN EXTREMAL WITH FREE AND UNFREE VARIATIONS AND TIME-DEPENDENT CONTROL BOUNDARIES.

$$dq_j [\tau_c(\epsilon), \epsilon] \Big|_{\epsilon = \epsilon_0} = q'_{jc} d\tau_c + \eta_{jc} d\epsilon; \quad q'_{jc} = q'_j(\tau_c, \epsilon_0); \quad \eta_c = \eta(\tau_c, \epsilon_0)$$

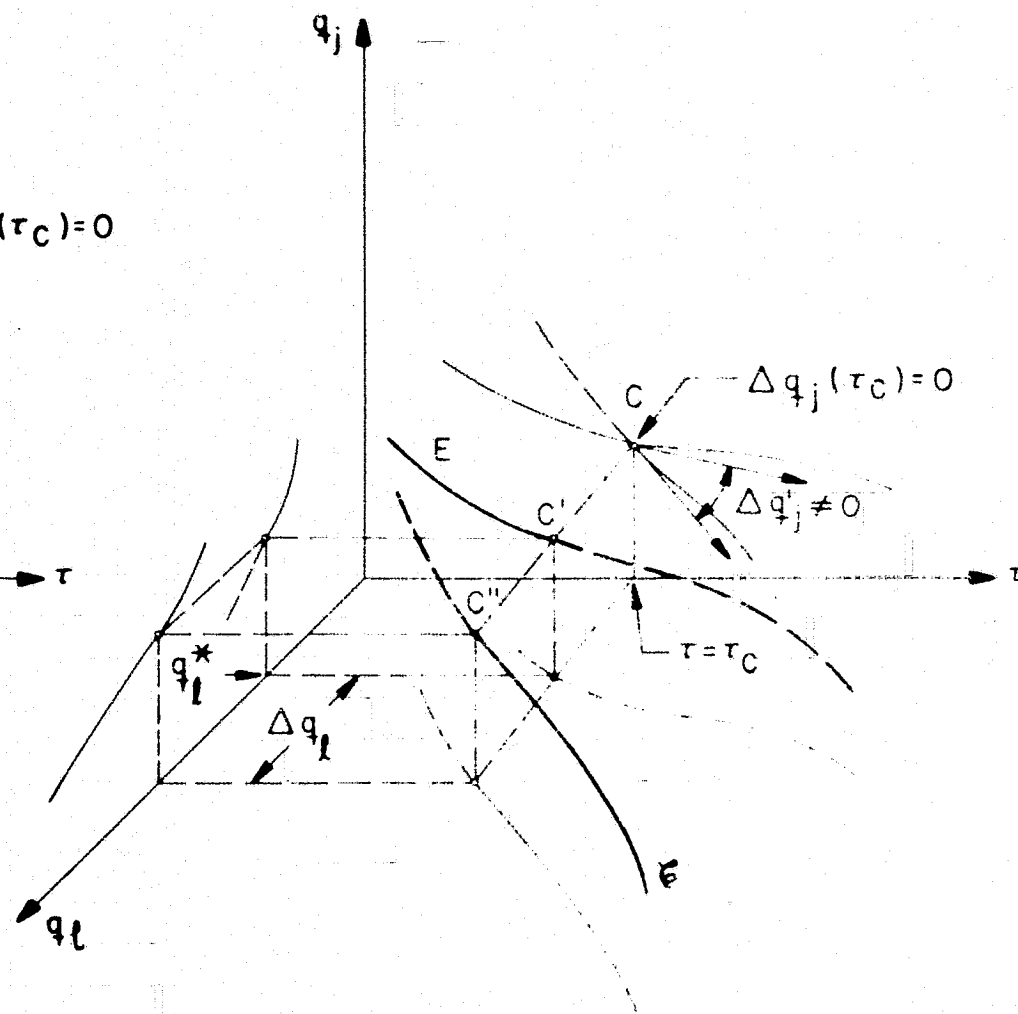


$$\eta_{c-0} = \eta_{c+0} \quad ; \quad dq_{jc-0} \neq dq_{jc+0}$$

FIGURE 5. CORNER CONTINUITY CONDITIONS IN THE  $\eta_j$  - VARIATIONS.

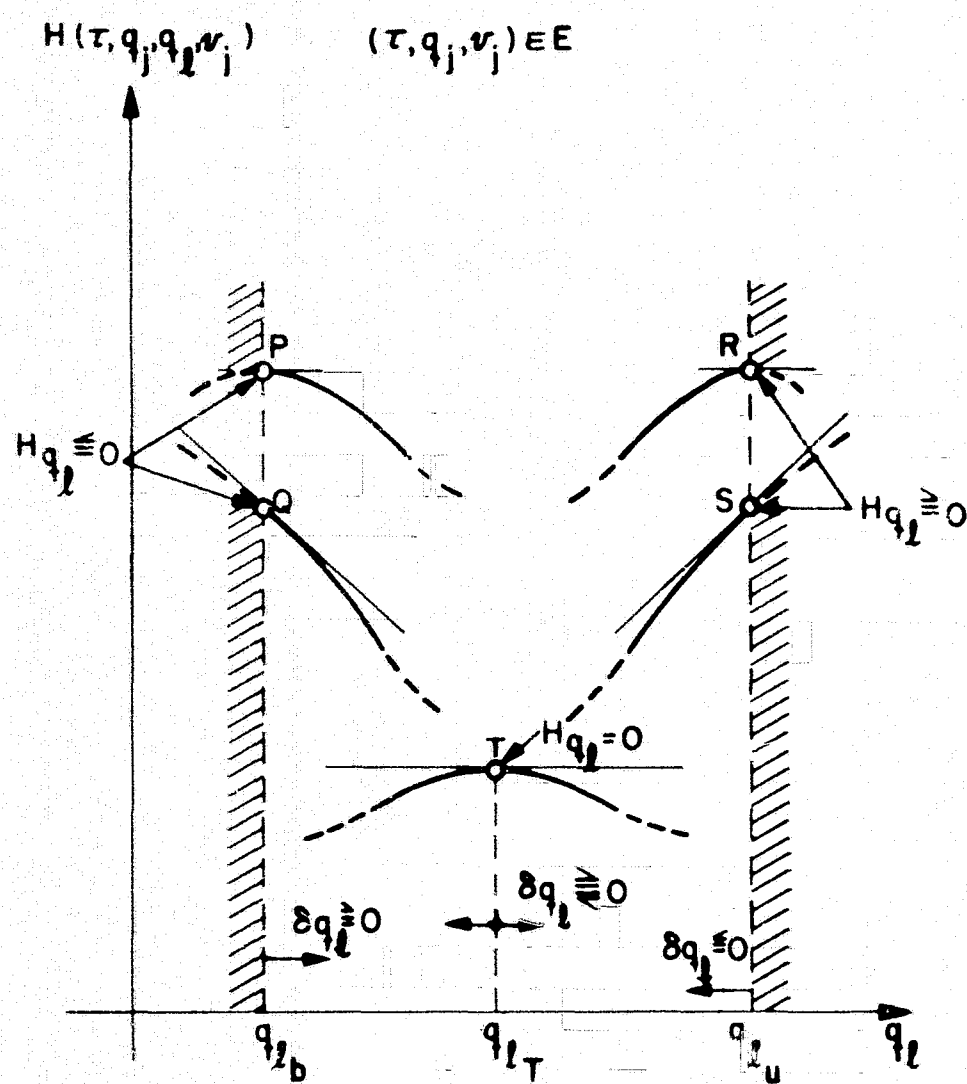


(a)

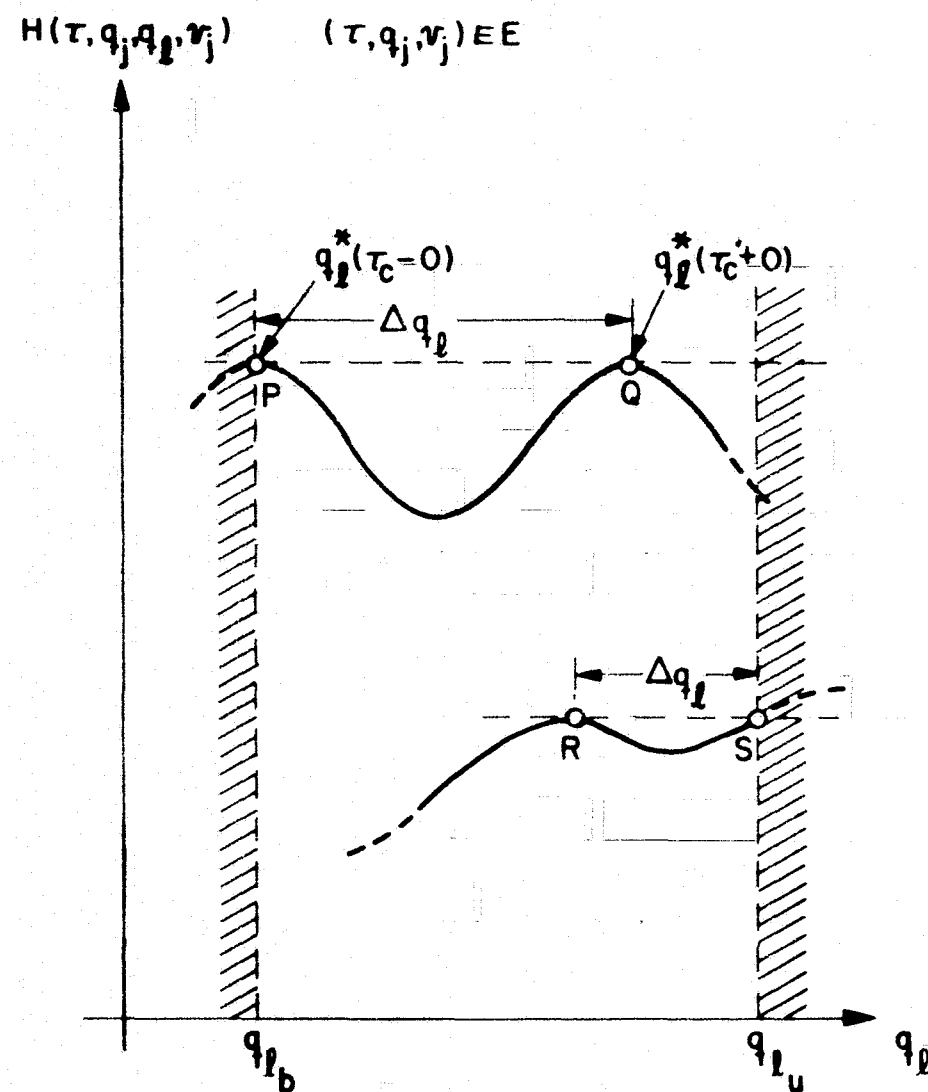


(b)

FIGURE 6. EXTREMAL ARC E AND DIFFERENTIABLE ADMISSIBLE COMPARISON ARC C IN THE  $(\tau, q_j, q_l, v_j)$  - SPACE OF CANONICAL VARIABLES.



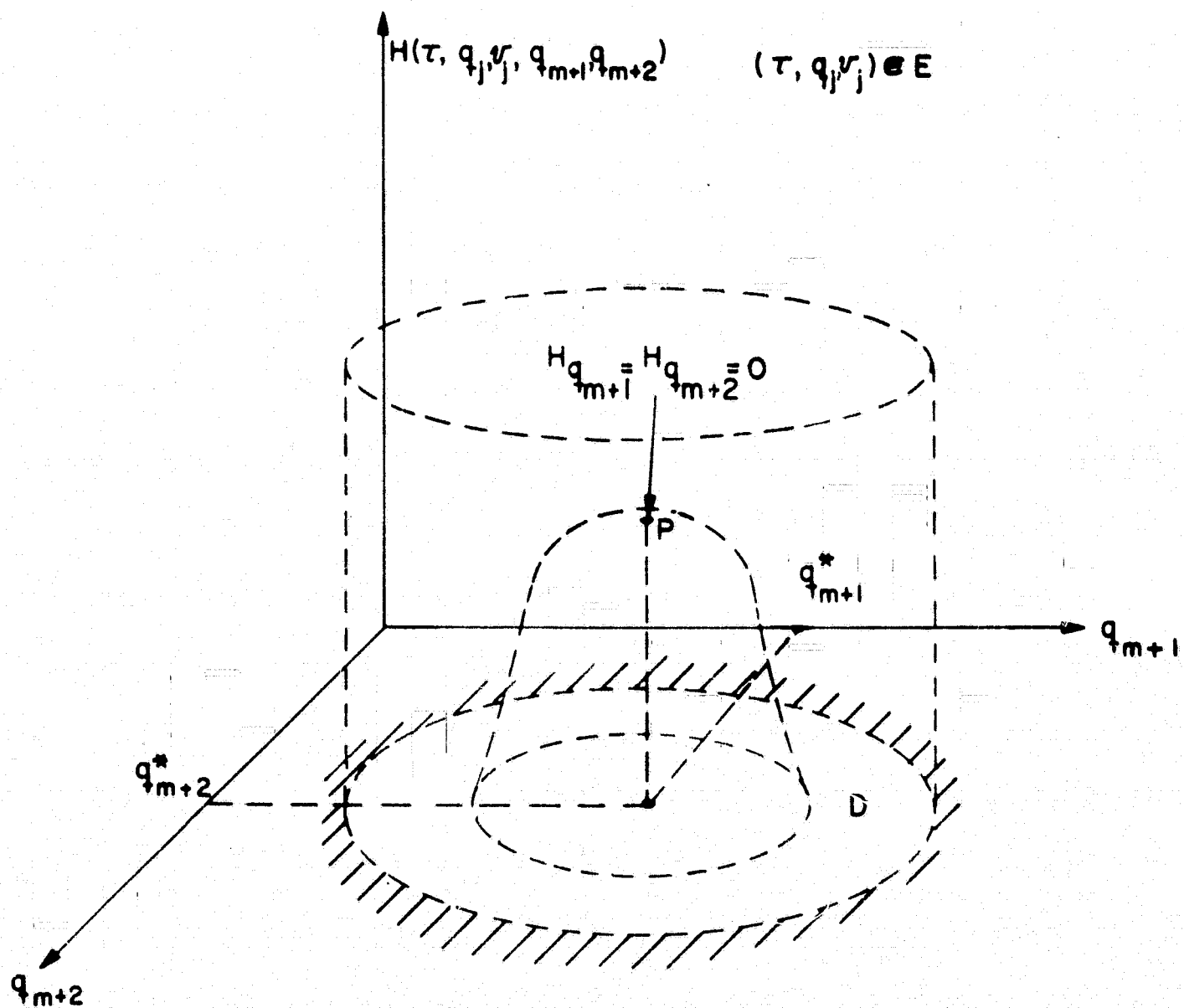
(a)



(b)

FIGURE 7. MAXIMALITY PRINCIPLE AND CORNER CONDITIONS REPRESENTATION IN THE  $(q, q_l)$  - PLANE.

# CASE OF MULTIPLE CONTROL VARIABLES



$D$  = REGION OF ADMISSIBLE CONTROL

FIGURE 3. MAXIMALITY PRINCIPLE FOR THE CASE OF MULTIPLE CONTROL.

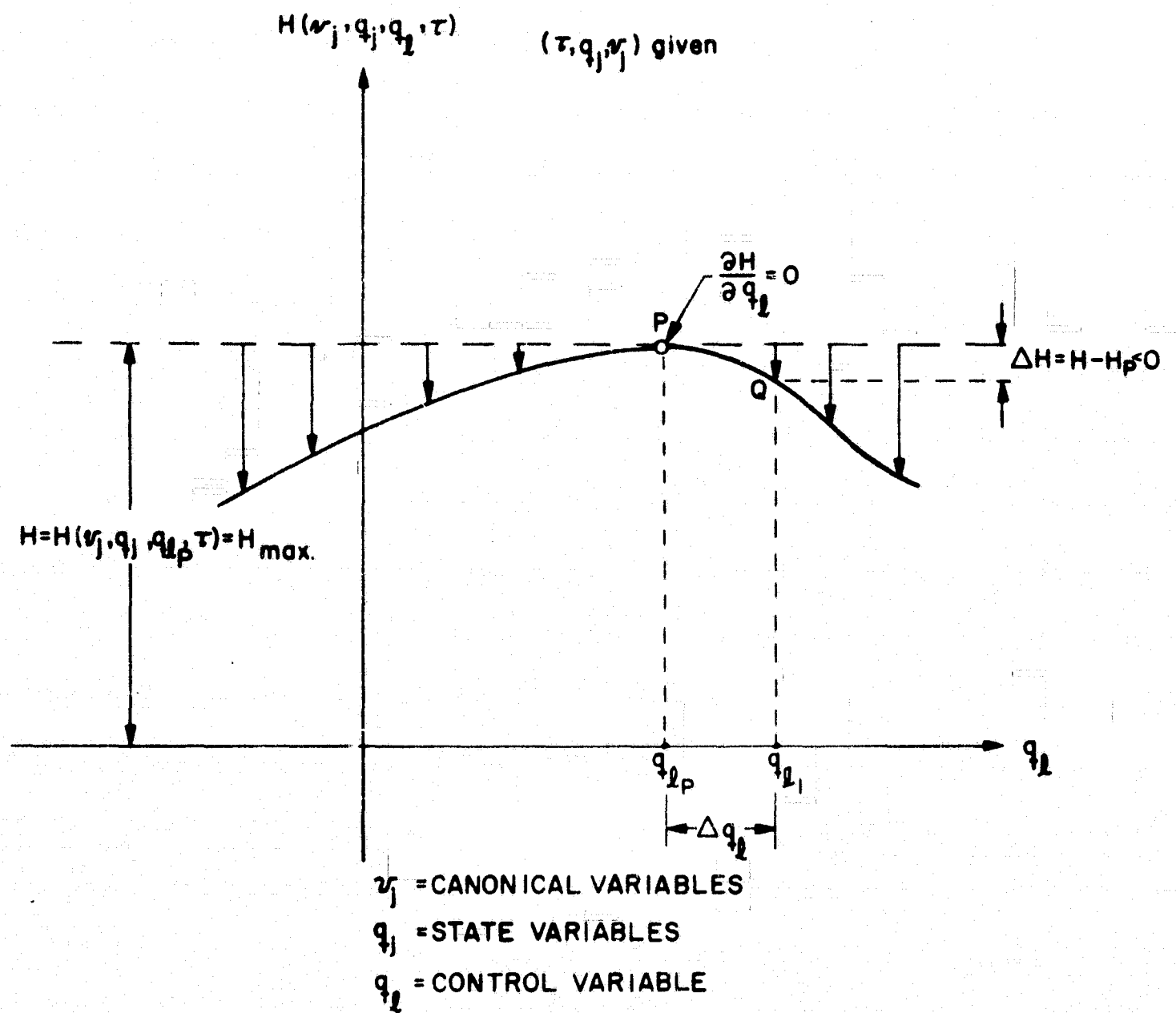
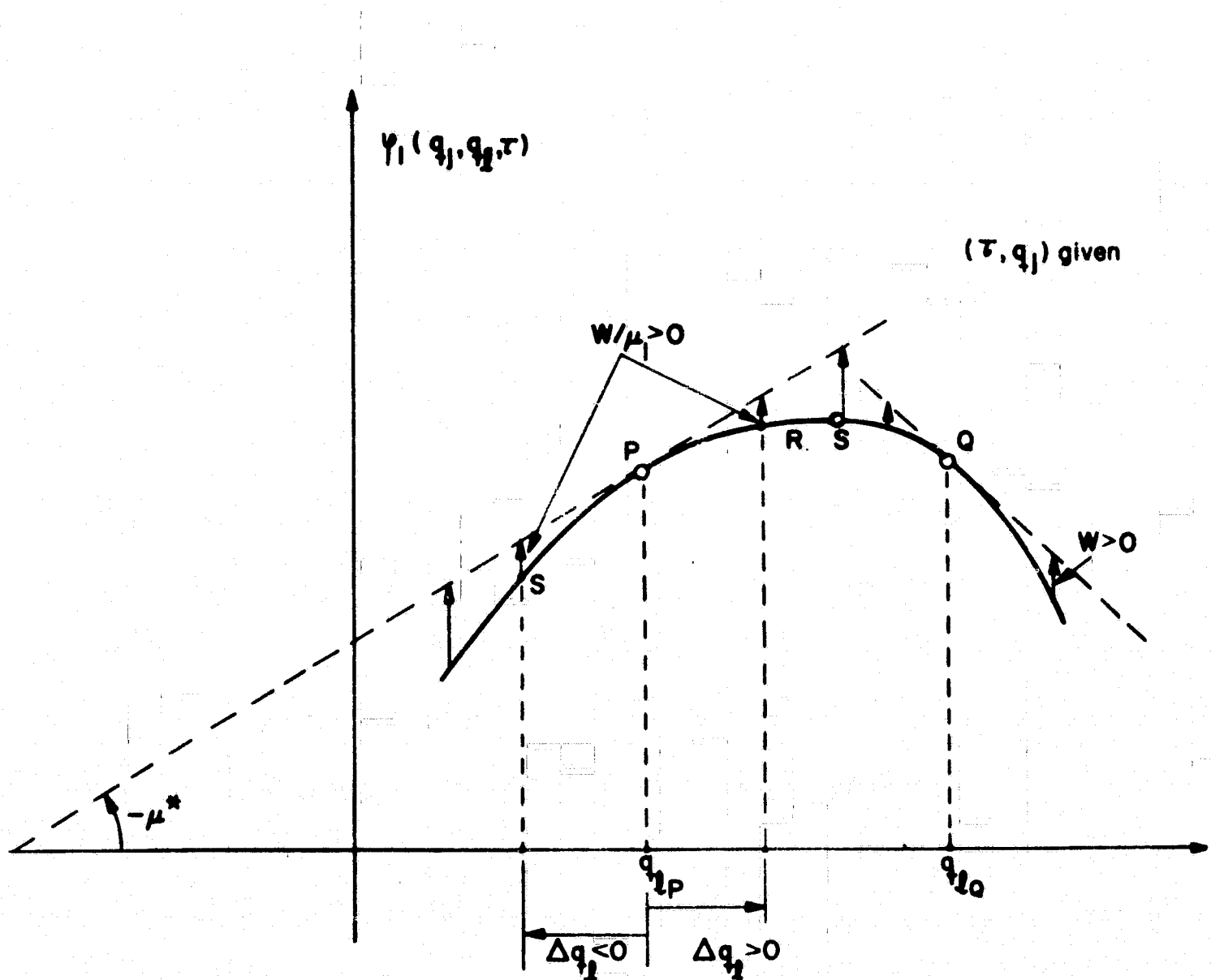
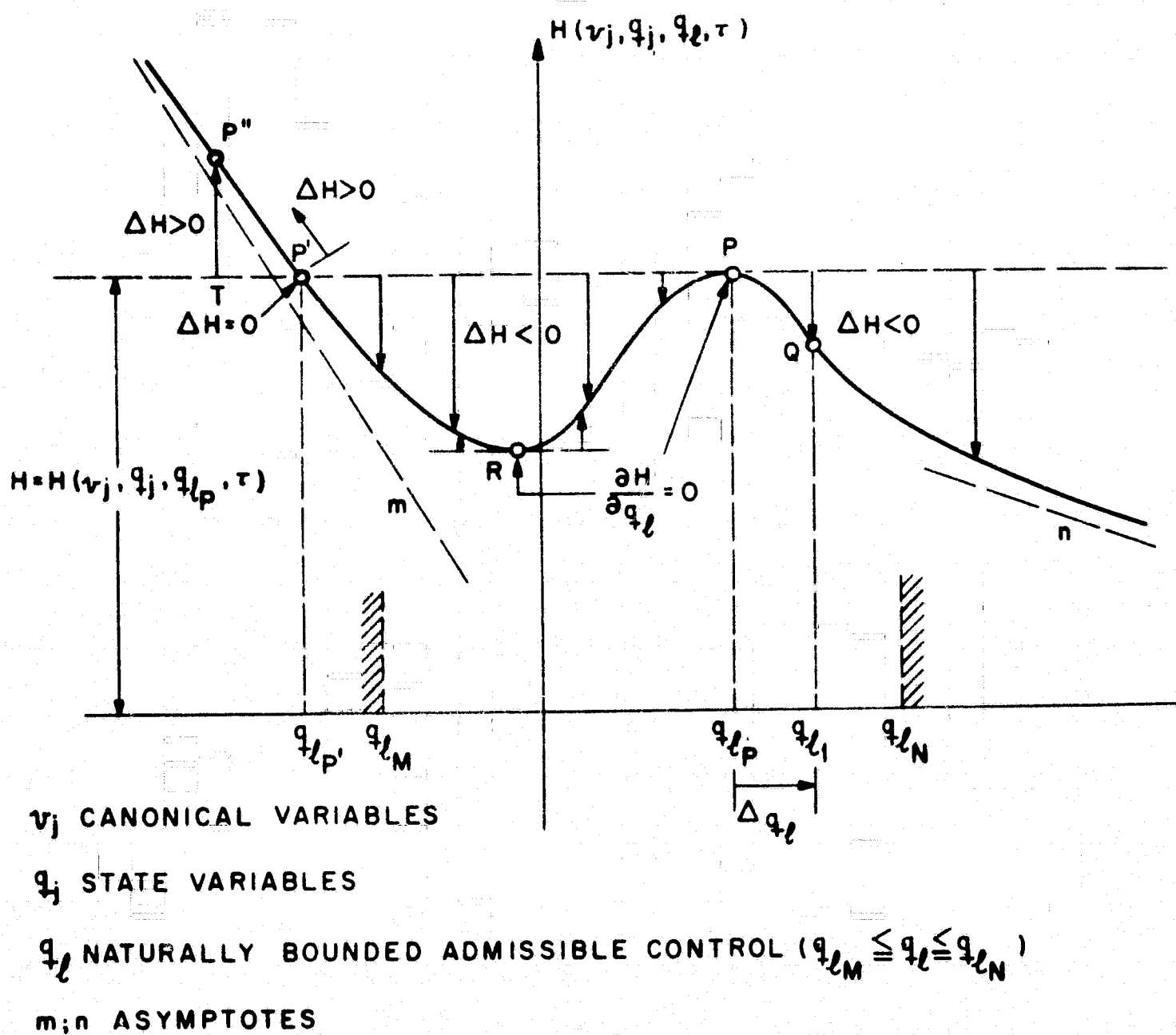


FIGURE 9. THE H-LINE FOR THE CASE OF UNBOUNDED CONTROL.



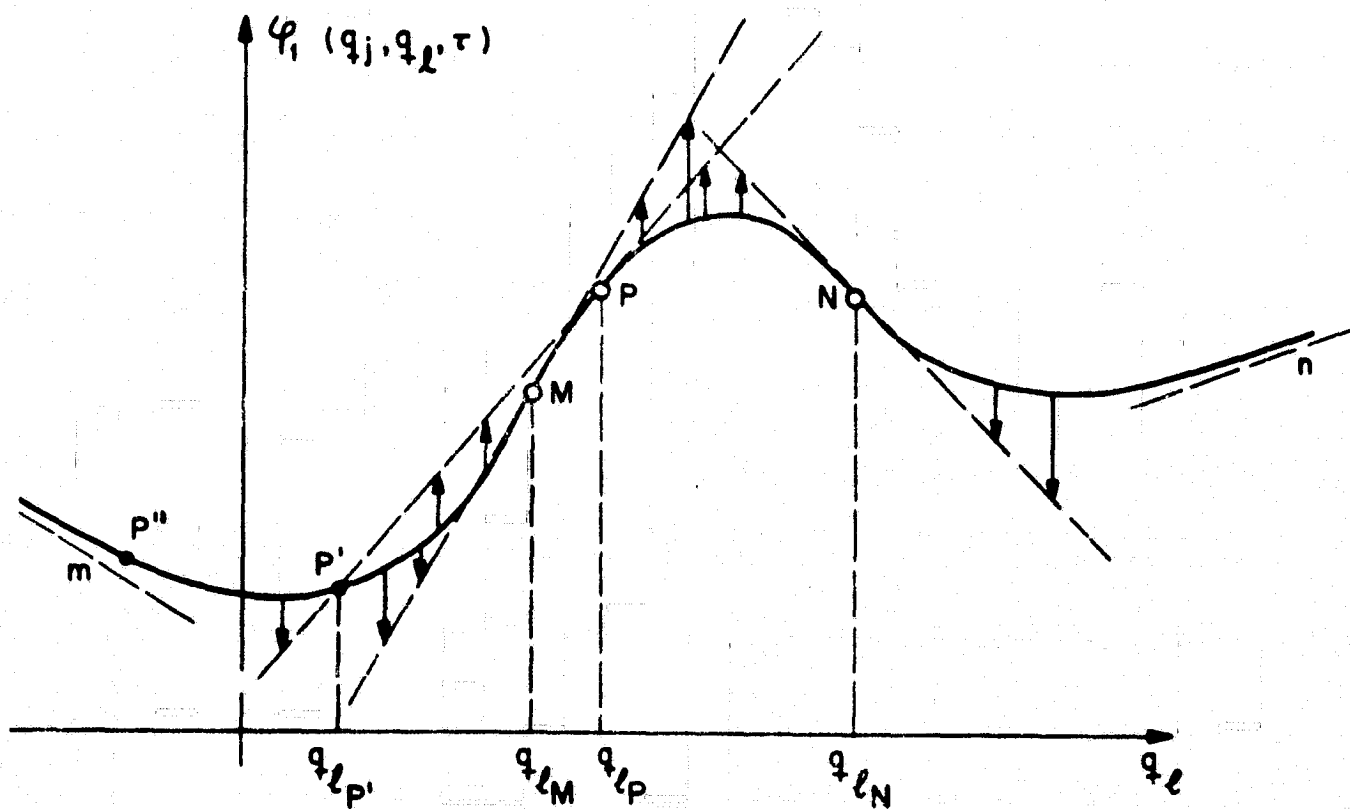
W-TEST SATISFIED AT ANY POINT ON THE CHARACTERISTIC  
 $q_1$  STATE VARIABLES  
 $q_2$  CONTROL VARIABLE  
 P; Q EXAMPLE OF ADMISSIBLE OPERATION POINTS

FIGURE 10. THE CHARACTERISTIC LINE FOR THE CASE OF UNBOUNDED CONTROL.



(a)

FIGURE 11. CASES OF NATURALLY BOUNDED CONTROL.



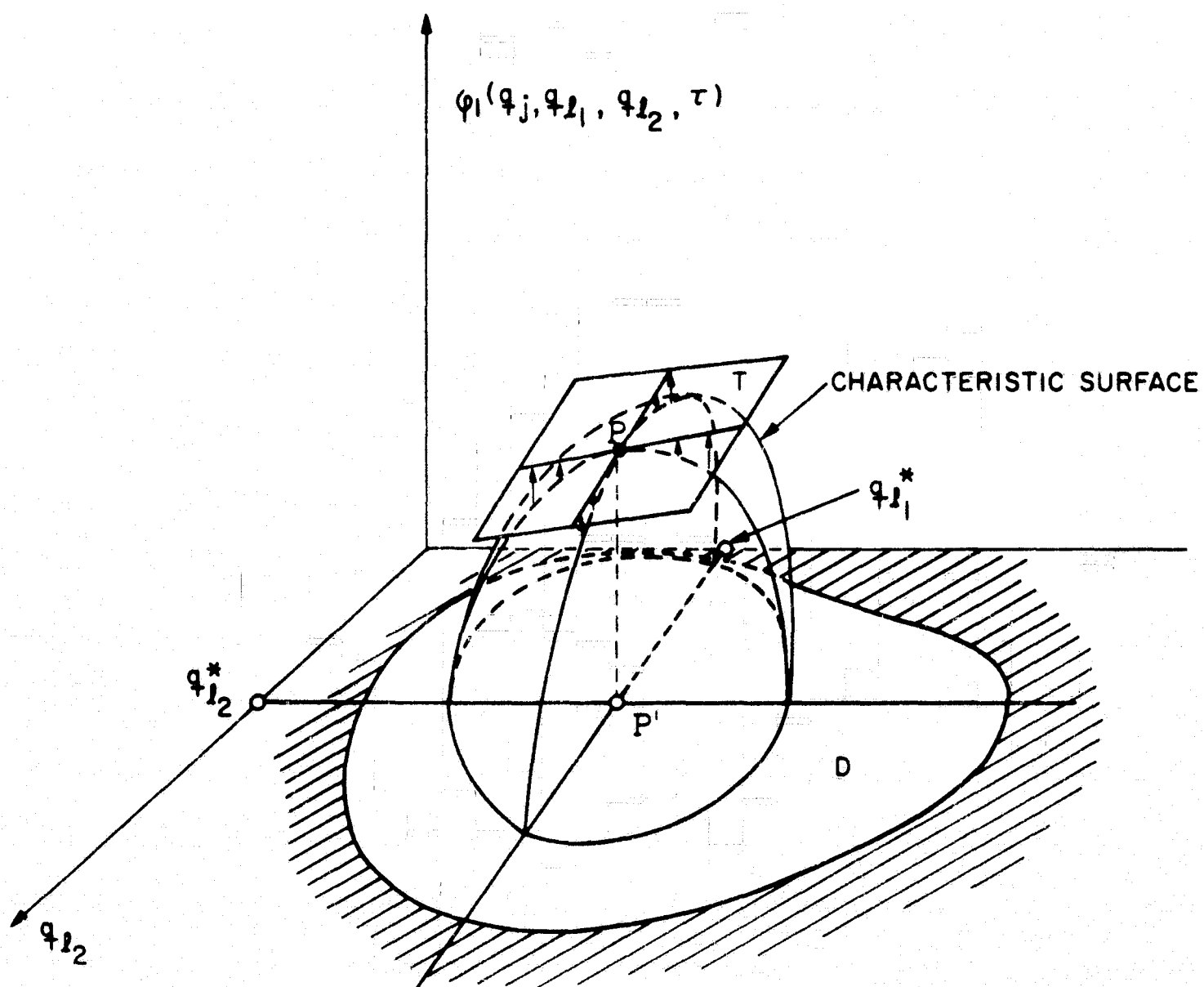
$q_j$  STATE VARIABLES ON  $E$

$q_l$  NATURALLY BOUNDED ADMISSABLE CONTROL ( $q_{lM} \leq q_l \leq q_{lN}$ )

$m; n$  ASYMPTOTES

(b)

FIGURE 11. CASES OF NATURALLY BOUNDED CONTROL (Cont'd)

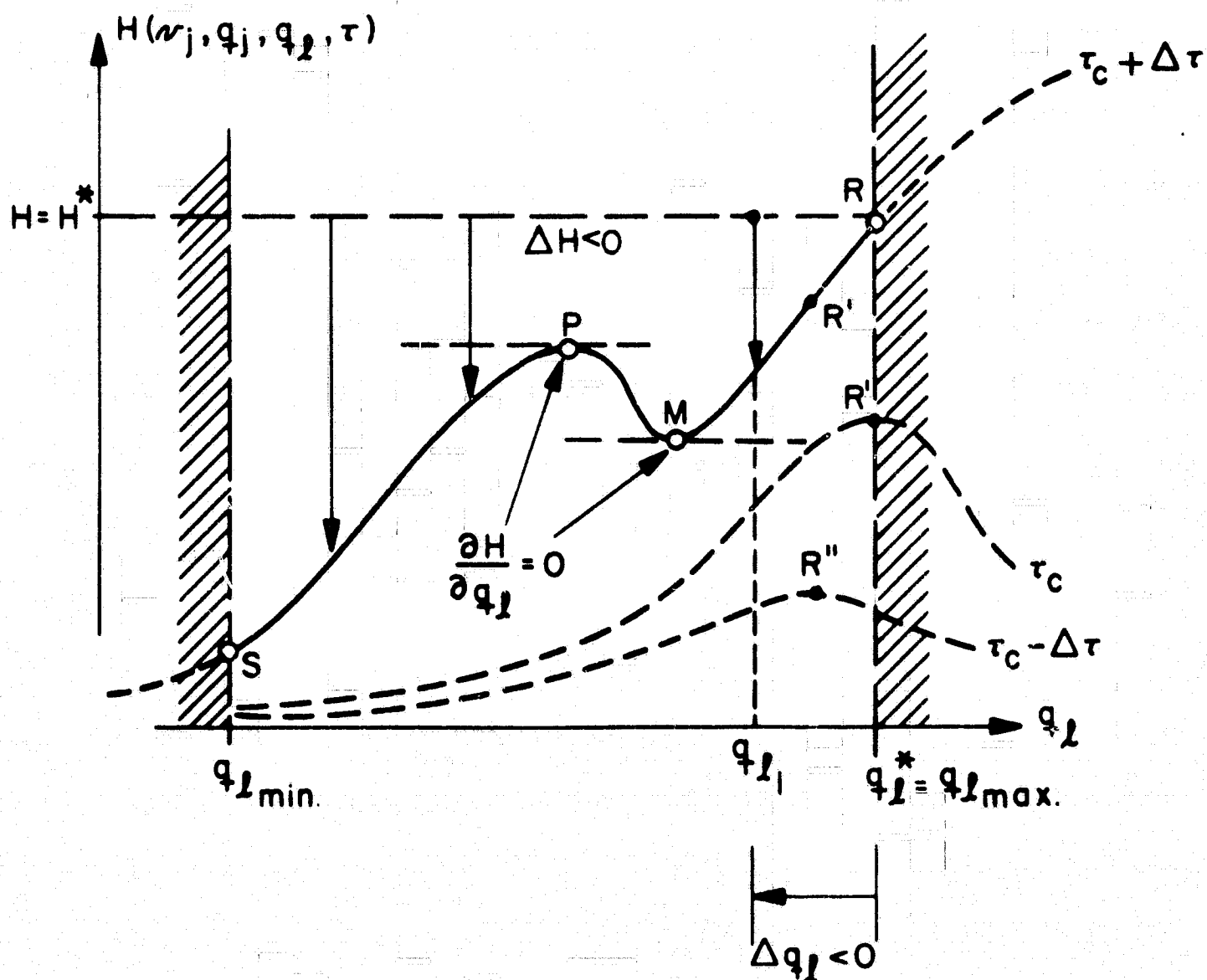


D = BOUNDED DOMAIN OF ADMISSIBLE CONTROL

T = TANGENT PLANE AT THE OPERATING POINT P

FIGURE 12. THE CHARACTERISTIC SURFACE FOR THE CASE OF MULTIPLE CONTROL AND THE WEIERSTRASS CONDITION.

$$[\tau, q_j(\tau), \nu_j(\tau)] \text{ given}$$



$q_j$  STATE VARIABLES

$q_l$  BOUNDED ADMISSIBLE CONTROL ( $q_{l \min.} \leq q_l \leq q_{l \max.}$ )

(a)

FIGURE 13. THE H-LINE AND THE CHARACTERISTIC LINE FOR CASES OF CONTROL ON THE BOUNDARY. CICALA'S POINTER REPRESENTATION.

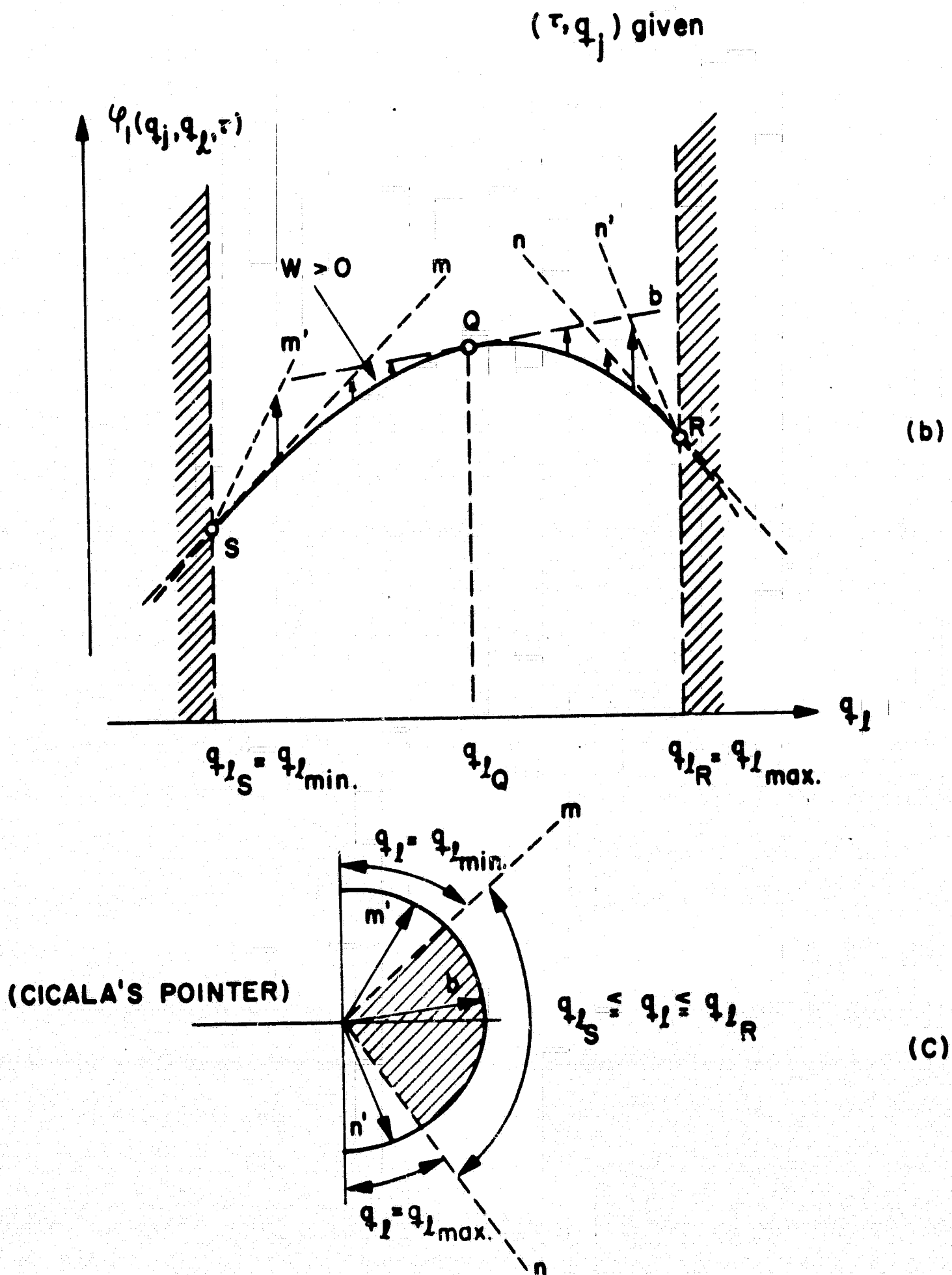


FIGURE 13. THE H-LINE AND THE CHARACTERISTIC LINE FOR CASES OF CONTROL ON THE BOUNDARY. CICALA'S POINTER REPRESENTATION. (Cont'd)

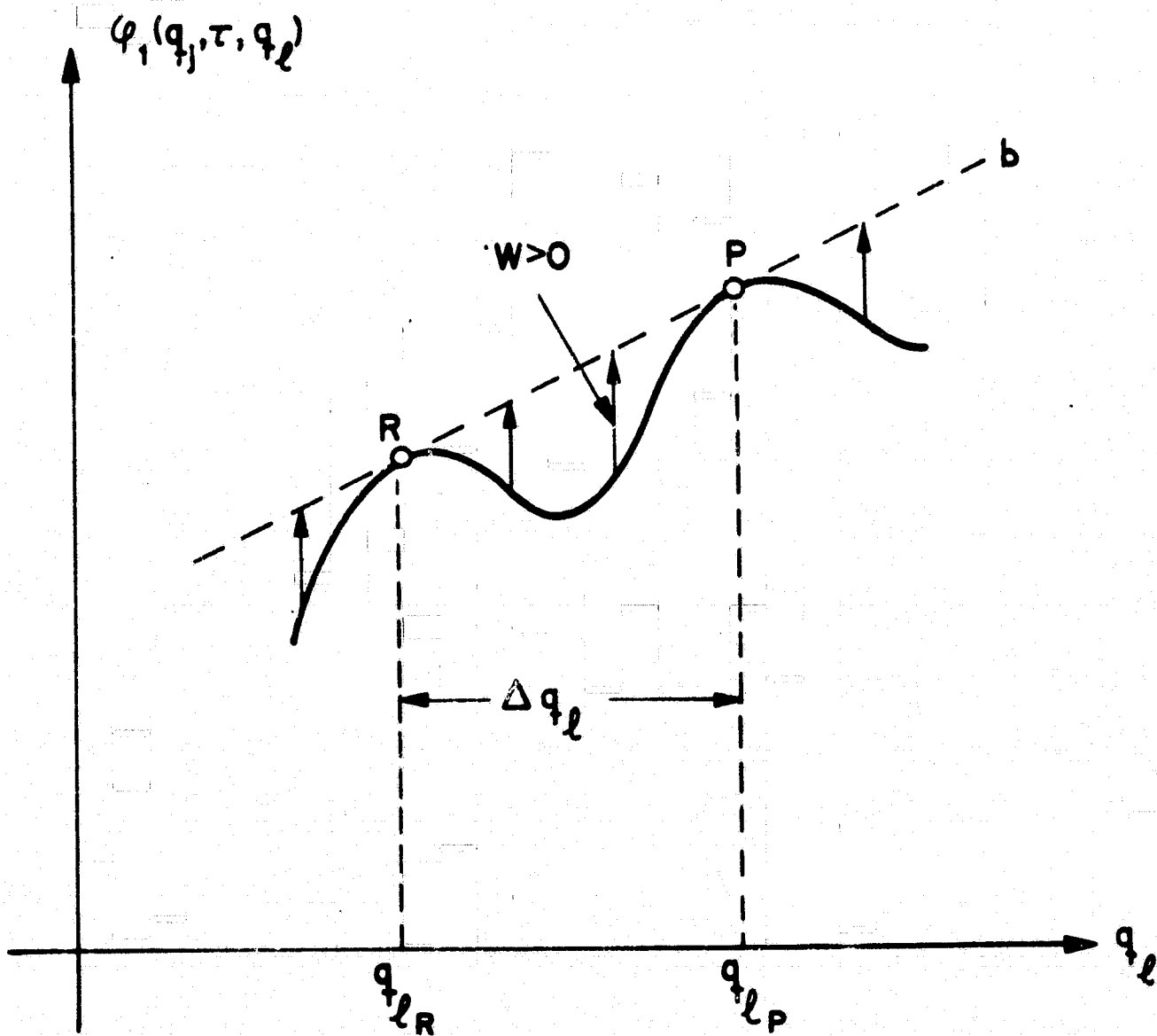


FIGURE 14. CORNER POINT WITH DISCONTINUOUS CONTROL

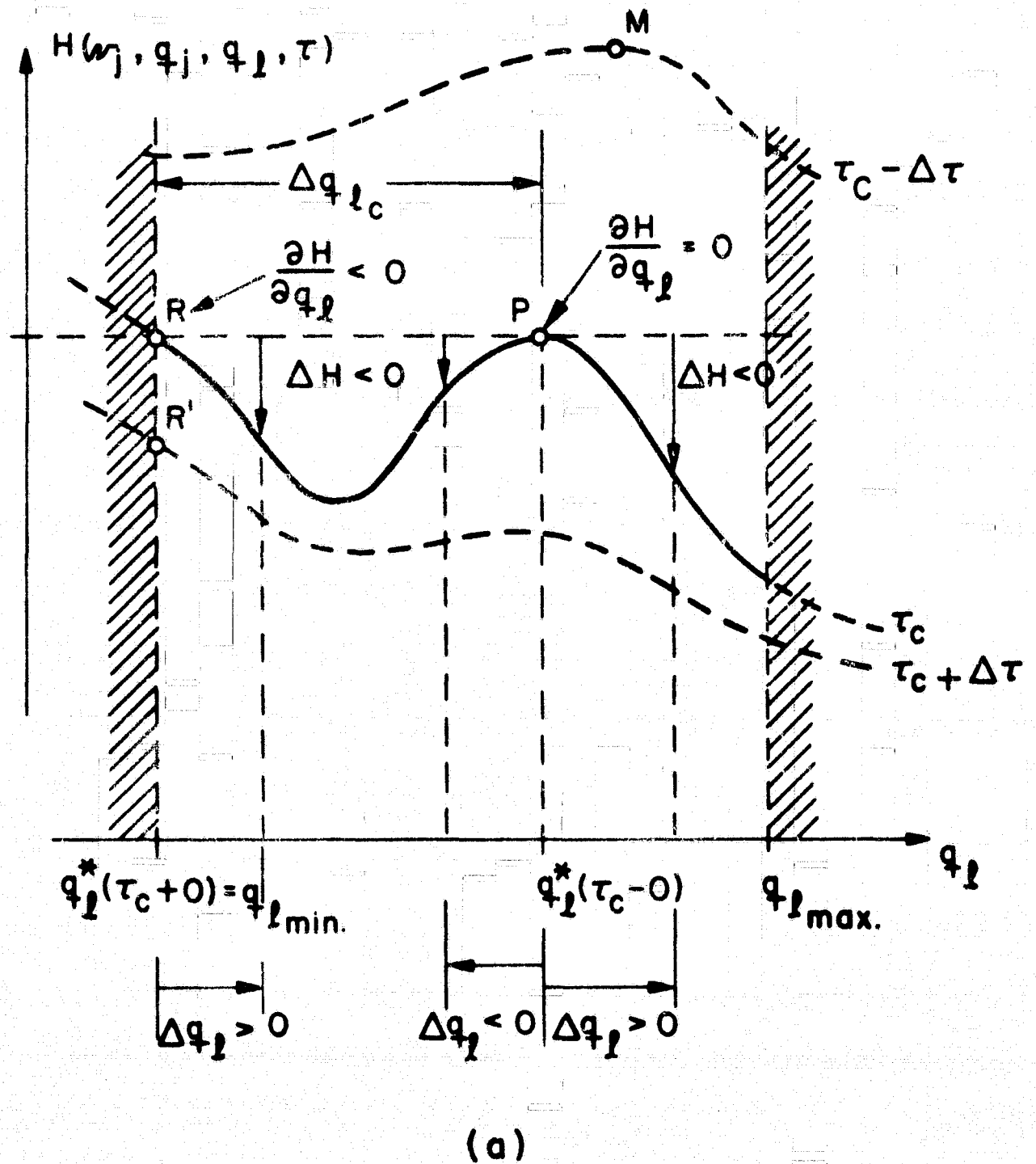
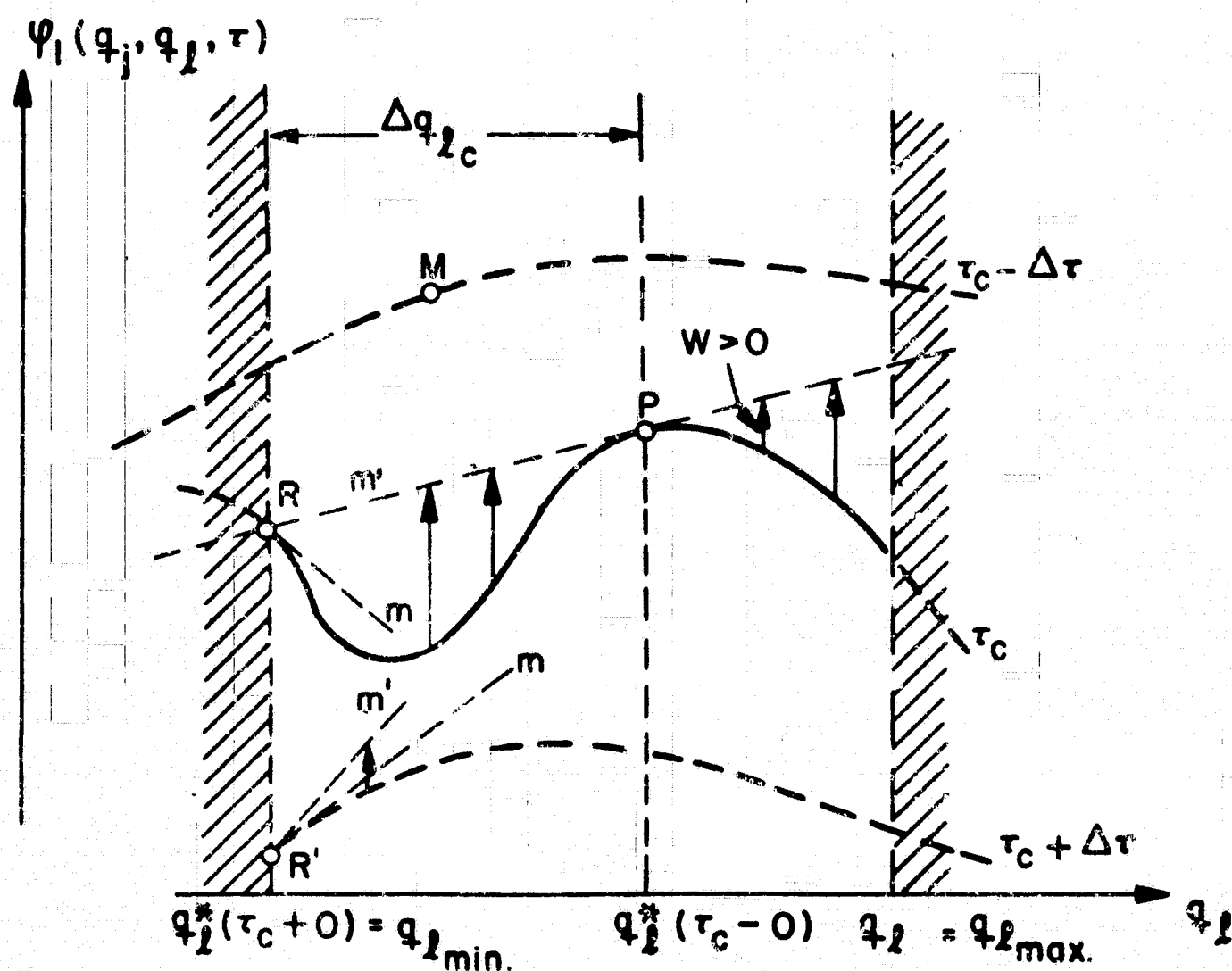


FIGURE 15. TIME DEPENDENT H-LINE AND CHARACTERISTIC LINE AT DIFFERENT TIMES ALONG A BROKEN EXTREMAL.



$q_j$  STATE VARIABLES

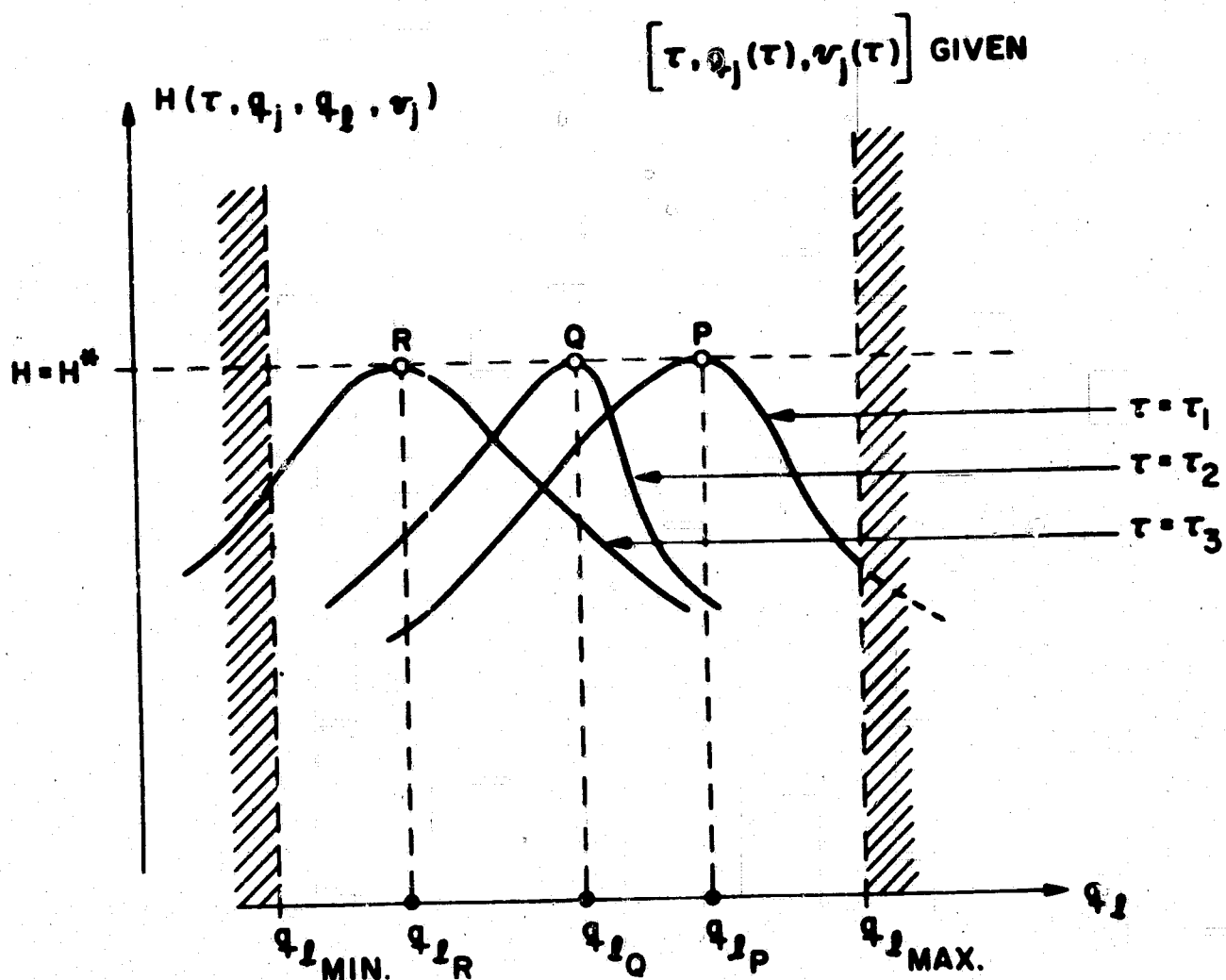
$q_l$  BOUNDED CONTROL VARIABLE  $\left[ q_{l_{min}}(\tau) \leq q_l(\tau) \leq q_{l_{max}}(\tau) \right]$

$q_l^*$  ADMISSIBLE OPTIMUM CONTROL

$\Delta q_{lc}$  ADMISSIBLE JUMP OR DISCONTINUOUS CONTROL AT CORNER C.

(b)

FIGURE 15. TIME DEPENDENT H-LINE AND CHARACTERISTIC LINE AT DIFFERENT TIMES ALONG A BROKEN EXTREMAL. (Cont'd)



$$H^* = H_P = H_Q = H_R = \text{CONST.}$$

$$q_1^*(\tau_1) = q_{1P}$$

$$q_1^*(\tau_2) = q_{1Q}$$

$$q_1^*(\tau_3) = q_{1R}$$

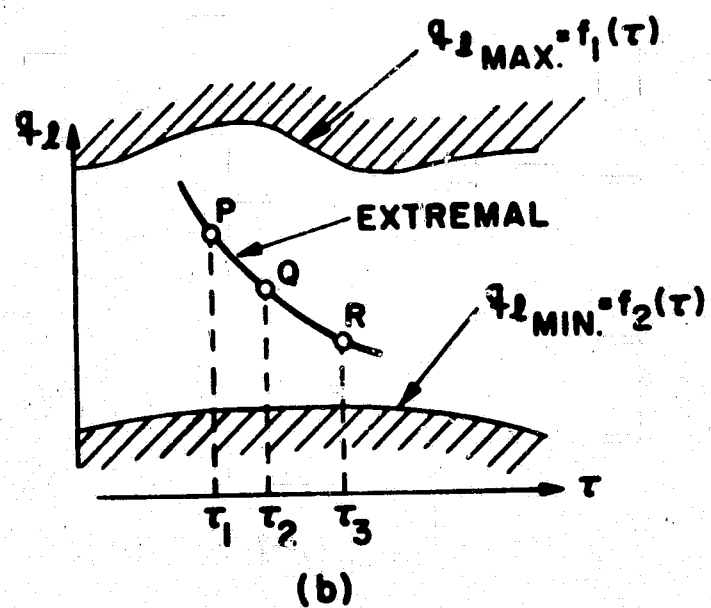
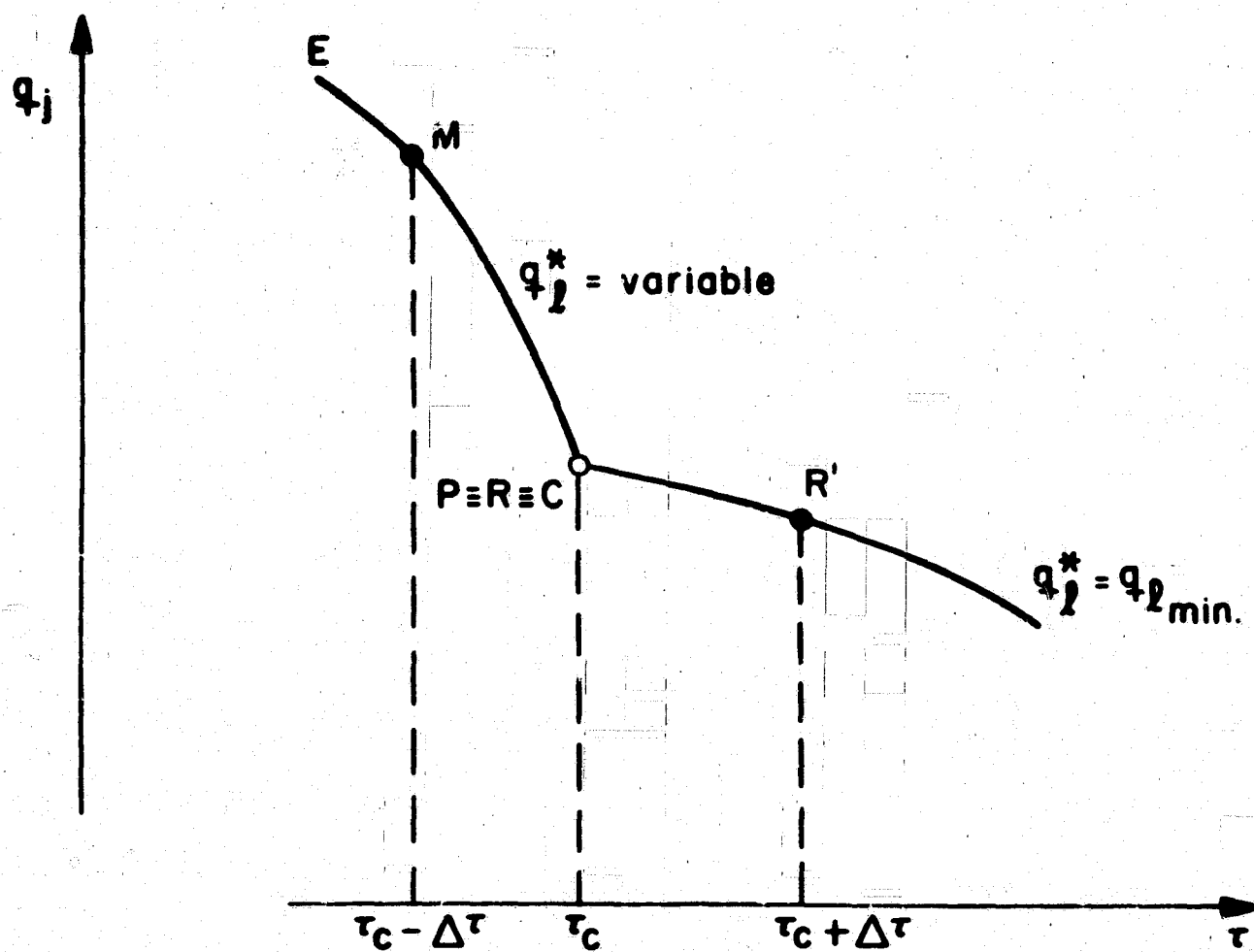


FIGURE 16. H-LINE AT DIFFERENT TIMES ALONG AN EXTREMAL FOR THE CASE  $\frac{\partial H}{\partial \tau} = 0$ .



(c)

FIGURE 16. H-LINE AT DIFFERENT TIMES ALONG A EXTREMAL FOR THE CASE  $\frac{\partial H}{\partial \tau} = 0$ .

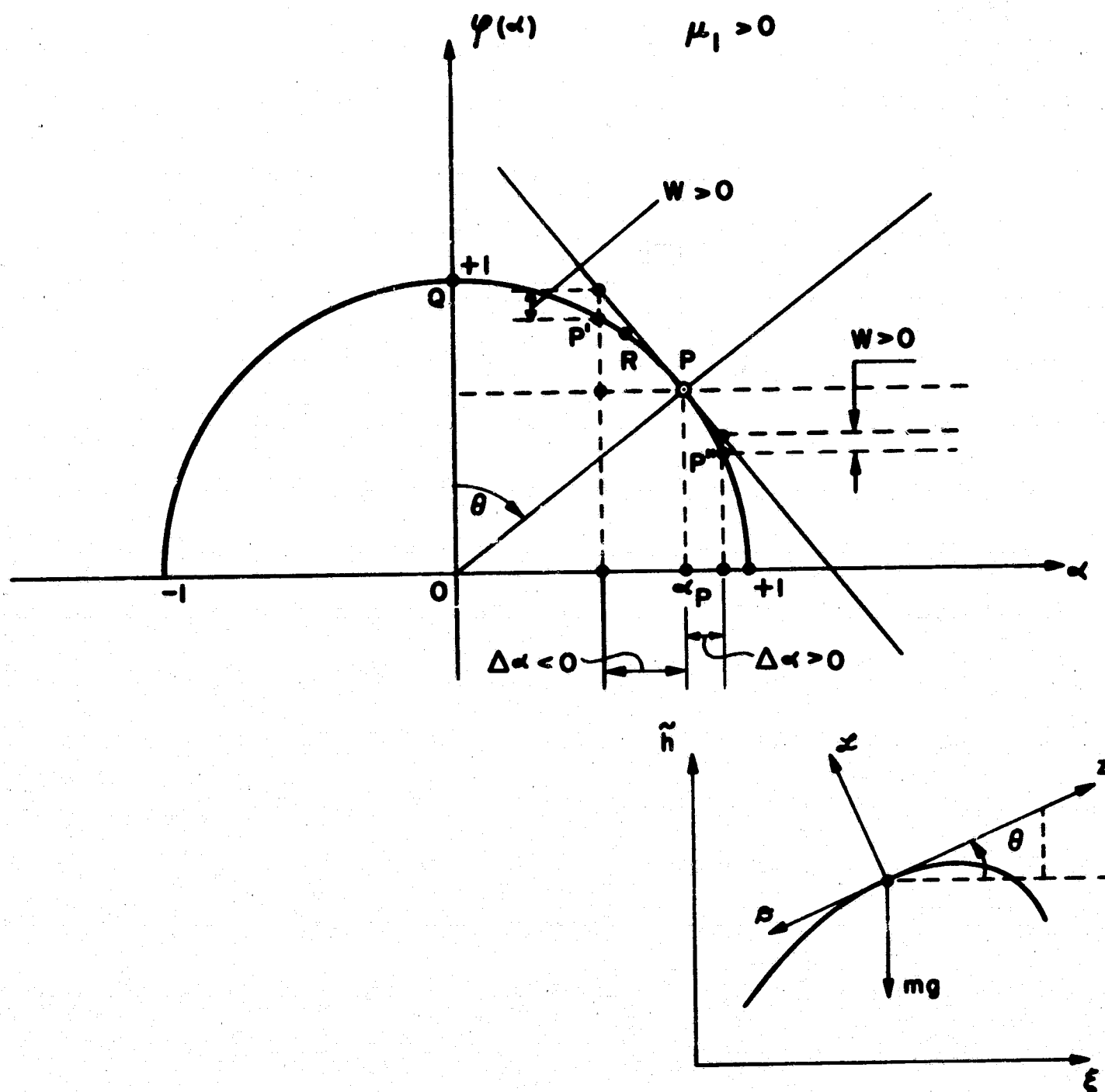
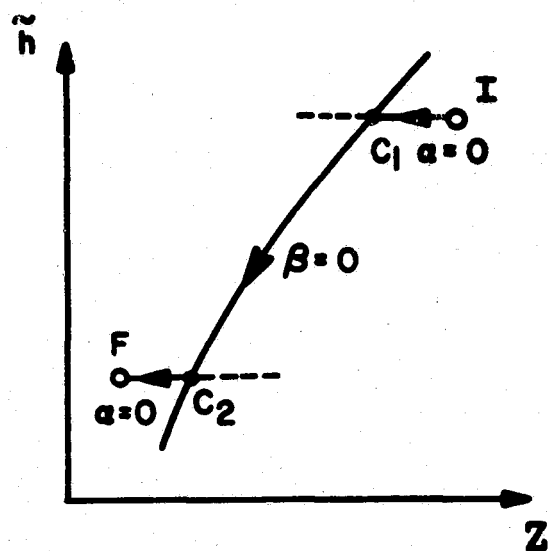
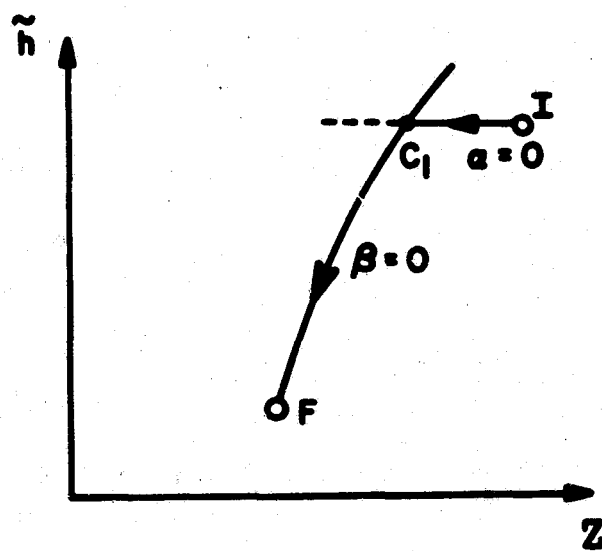


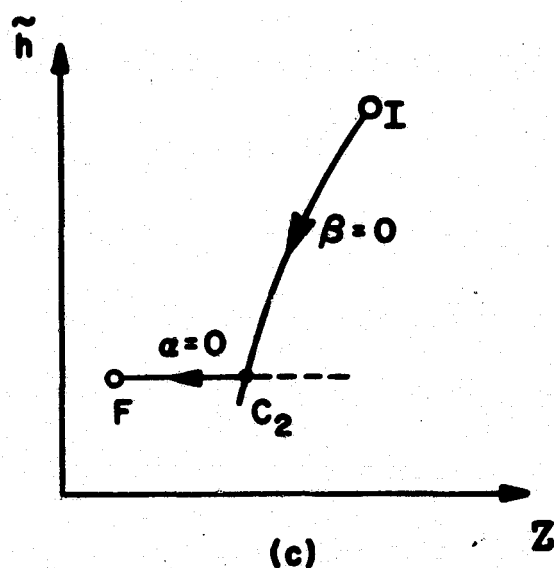
FIGURE 17. CHARACTERISTIC LINE FOR EXAMPLE I.



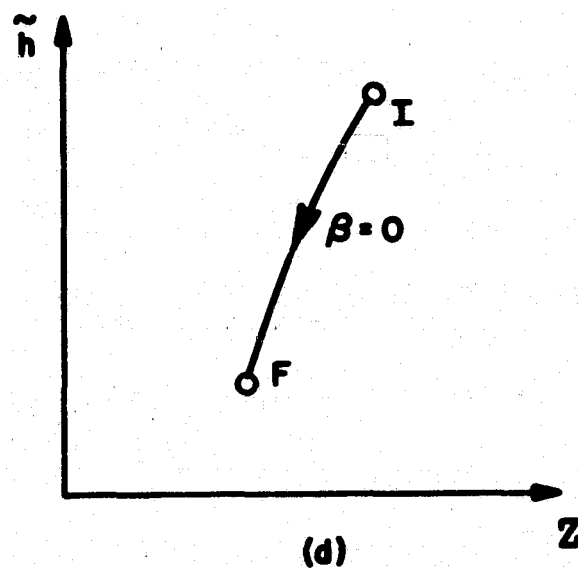
(a)



(b)



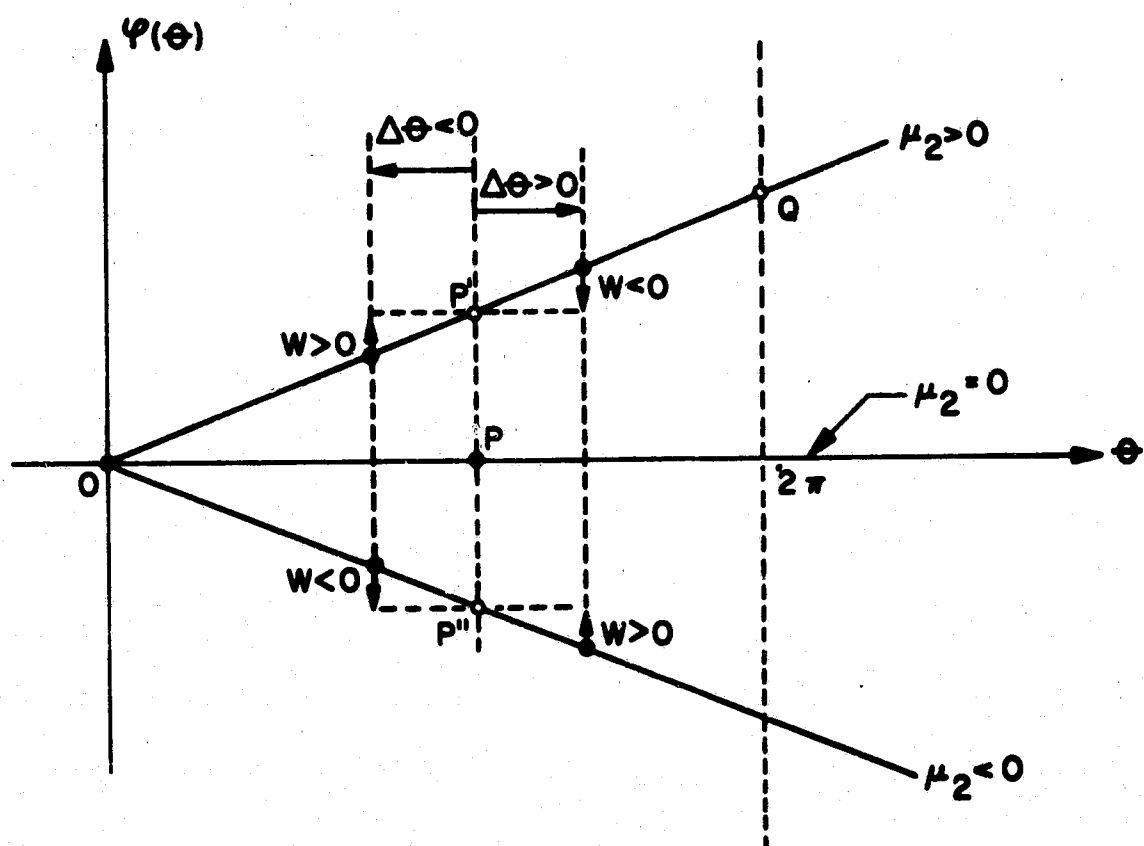
(c)



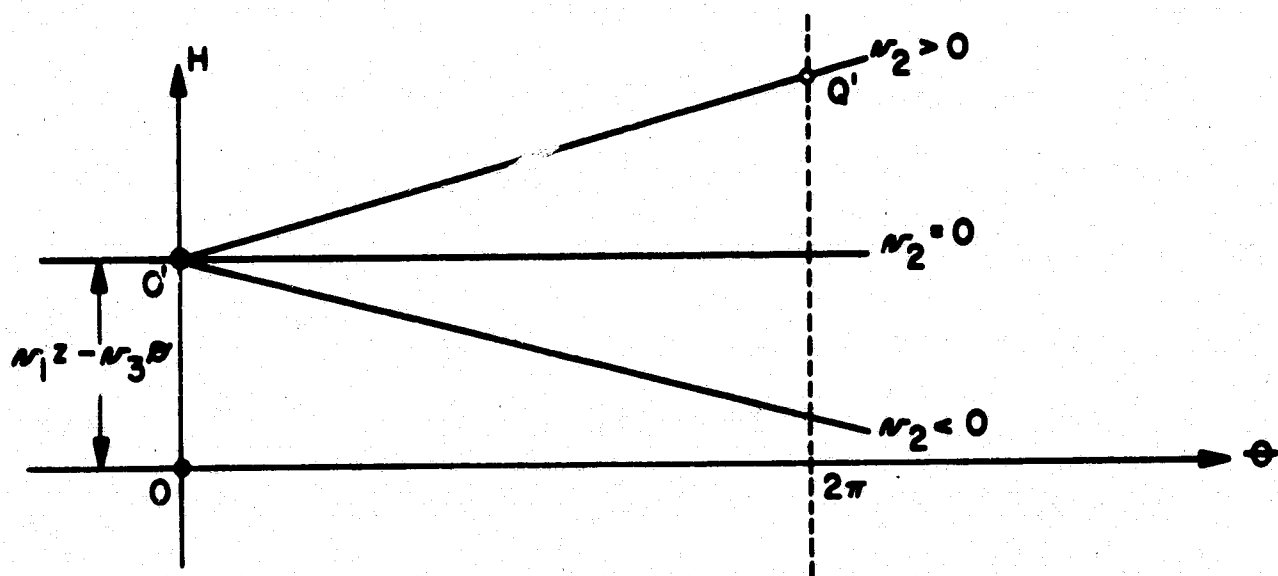
(d)

I, F, INITIAL AND FINAL POINTS RESPECTIVELY

FIGURE 18. DIFFERENT BOUNDARY VALUE PROBLEMS, AND SUB-ARCS FORMING THE EXTREMAL SOLUTION FOR EXAMPLE I.

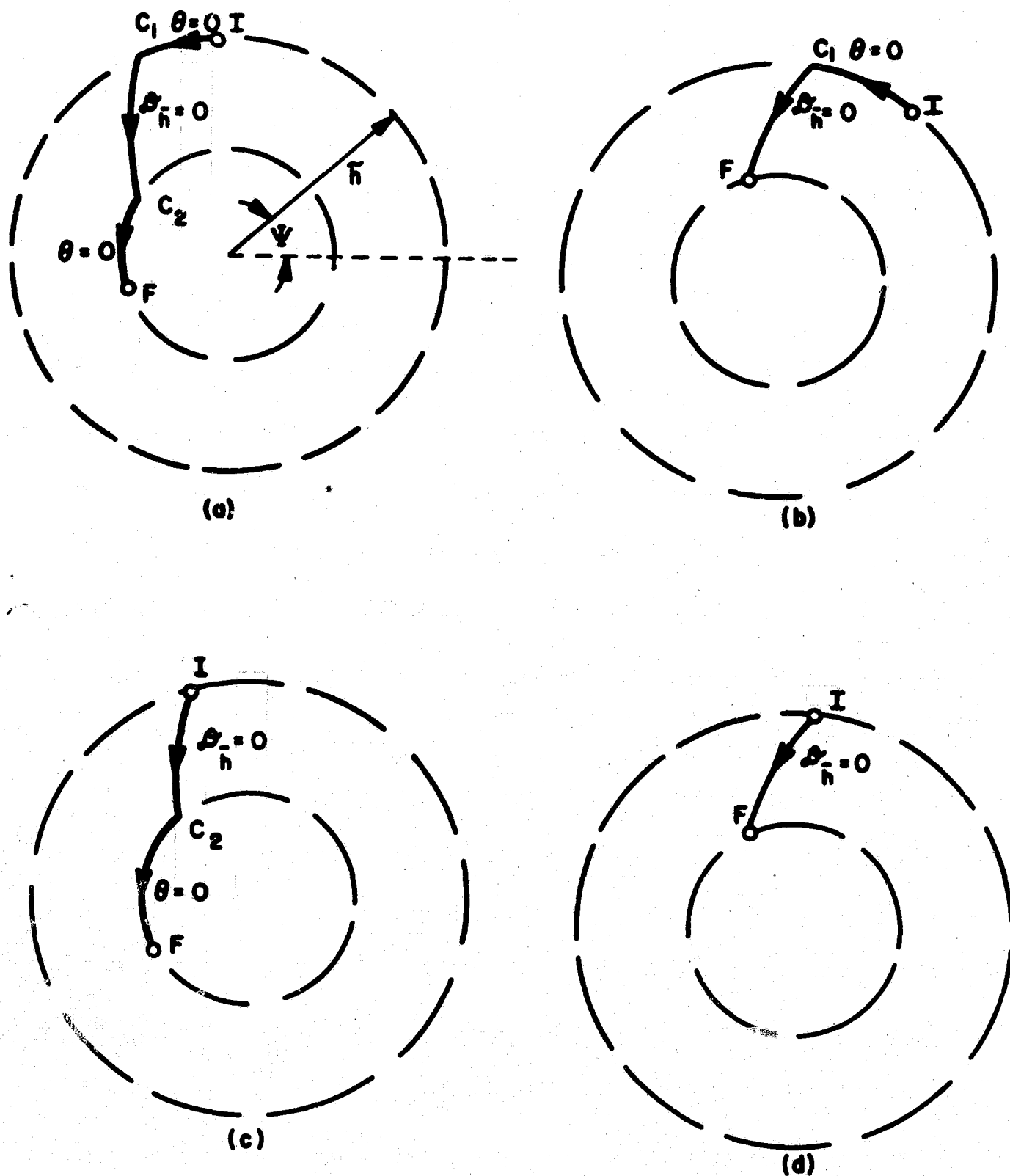


(a)



(b)

FIGURE 19. CHARACTERISTIC LINES AND H-LINES FOR EXAMPLE II.



I, F, INITIAL AND FINAL POINTS RESPECTIVELY

FIGURE 20. DIFFERENT BOUNDARY VALUE PROBLEMS, AND SUB-ARCS FORMING THE OPTIMUM TRANSFER FOR EXAMPLE II.

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**REPUBLIC AVIATION CORPORATION**

**THE HAMILTON-JACOBI FORMULATION OF THE  
RESTRICTED THREE BODY PROBLEM IN  
TERMS OF THE TWO FIXED CENTER PROBLEM**

**By**

**Mary Payne  
Samuel Pines**

**Farmingdale, L. I., New York**

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# DEFINITION OF SYMBOLS

$\underline{R}$	Position vector of the vehicle relative to the barycenter in a coordinate system fixed in space
$\underline{R}_1$	Position vector of the vehicle relative to the earth
$\underline{R}_2$	Position vector of the vehicle relative to the moon
$R'$	Position vector of the vehicle relative to the barycenter in a rotating system
$\bar{R}$	Position vector of the vehicle relative to the midpoint of the earth-moon line
$r_1$	Magnitude of $\underline{R}_1$
$r_2$	Magnitude of $\underline{R}_2$
$\Omega$	Angular velocity vector of earth-moon system
$\omega$	Magnitude of $\omega$
$\mu$	Gravitational constant times mass of the earth
$\mu'$	Gravitational constant times mass of the moon
$\mathcal{L}$	Lagrangian function
$\underline{P}$	Momentum vector
$H$	Hamiltonian function
$q_i$	Generalized coordinates conjugate to $p_i$
$Q_i$	Generalized coordinates conjugate to $P_i$
$p_i$	Generalized momenta conjugate to $q_i$
$P_i$	Generalized momenta conjugate to $Q_i$

## DEFINITION OF SYMBOLS (Cont'd)

$\psi_1$	Time-dependent generating function
$t$	Time
$H_1$	Integrable part of the Hamiltonian
$H_2$	Disturbing function
$h$	Energy constant for $H_1$
$W$	Time-independent generating function
$J_i$	Action variables
$w_i$	Angle variables
$\nu_i$	Frequencies for two fixed center problem
$q_1$ )	Elliptic coordinates )
$q_2$ )	) prolate spheroidal
	) coordinates
$\phi$	Angle measured around x-axis )
$c$	Half the distance between earth and moon
$P_1$ )	
$P_2$ )	Momenta conjugate to prolate spheroidal coordinates
$P_\phi$ )	
$x$ )	Rectangular coordinates in a system with earth at $(c, 0, 0)$ , moon
$y$ )	at $(-c, 0, 0)$ and $\underline{\Omega}$ in the z direction
$z$ )	
$\alpha$	Angular momentum about the line of centers in the two fixed center problem
$\beta$	Third dynamical constant of motion of the two fixed center problem
$R^2(q_1)$	Fundamental quartic associated with $q_1$
$S^2(q_2)$	Fundamental quartic associated with $q_2$
$u$	Parameter in terms of which coordinates and time of the two fixed center problem are given

## DEFINITION OF SYMBOLS (Cont'd)

$r_i$	Roots of $R^2(q_1) = 0$
$s_i$	Roots of $S^2(q_2) = 0$
$n_i$	Coefficient of linear term in $q_i$ contribution to time as a function of $u$
$m_i$	Coefficient of linear term in $q_i$ contribution to $\varphi$ as a function of $u$
$F_i(u)$	Periodic term in time due to $q_i$
$G_i(u)$	Periodic term in $\varphi$ due to $q_i$
$K_1$	Quarter period of $q_1$ elliptic functions
$K_2$	Quarter period of $q_2$ elliptic functions
$Q_h$	Coordinate conjugate to $h$
$Q_\alpha$	Coordinate conjugate to $\alpha$
$Q_\beta$	Coordinate conjugate to $\beta$

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**THE HAMILTON-JACOBI FORMULATION OF  
THE RESTRICTED THREE BODY PROBLEM  
IN TERMS OF THE TWO FIXED CENTER PROBLEM****By****Mary Payne****Samuel Pines****Summary**

19694

This report contains a development of the classical Hamilton-Jacobi perturbation techniques, applying the known solution of the Two Fixed Center Problem to the Restricted Three Body Problem.

**SECTION I. INTRODUCTION**

This report contains an outline of the development of a perturbation procedure for solving the restricted three body problem, using the solution of the two fixed center problem as an intermediate orbit. In the restricted problem, it is assumed that the two primary bodies move in circles about their center of mass, the barycenter. The primary bodies will be fixed in a coordinate system rotating with their angular velocity, so that the use of the two fixed center problem is immediately suggested. The two fixed center problem was first treated by Euler, who discovered that its equations of motion are separable in prolate spheroidal coordinates. A very complete discussion of the two fixed center problem has been given by Charlier<sup>(1)</sup>. This treatment covers some of the same ground as this report. It is from the

Hamiltonian point of view and includes a discussion of the action and angle variables, and the way in which the two fixed center problem would be used as a basis for a perturbation theory for the restricted problem. The only thing missing from Charlier's treatment is an explicit solution of the two fixed center problem, which would be necessary for the actual application to the restricted problem. Formal expressions for the action and angle variables are obtained from a more modern point of view by Buchheim<sup>(2)</sup>. Brief discussions of the two fixed center problem are given in many standard text books such as Whittaker<sup>(3)</sup>, Landau and Lifschitz<sup>(4)</sup> and Wintner<sup>(5)</sup>. The explicit solution of the two fixed center problem has been obtained by Pines and Payne<sup>(6)</sup>. In the present report, this solution will be combined with a Hamiltonian development of the problem to show how perturbation equations for the restricted problem may be obtained. A different development has been carried out recently by Davidson and Schulz-Arenstorff<sup>(7)</sup>. In this theory, the initial conditions of a two fixed center problem are used as parameters and a first order correction for the restricted problem is obtained in closed form. Second-order corrections are obtained by a numerical curve-fitting scheme.

In this report, Section II will contain a discussion of the restricted problem, and the way in which the two fixed center problem will be used. In Section III, the solution of the two fixed center problem will be outlined in sufficient detail for the determination of the action and angle variables, which is carried out in Sections IV and V. Finally in Section VI a summary will be given of the essential steps still necessary to obtain the solution of the restricted problem.

## SECTION II - THE RESTRICTED PROBLEM

The equations of motion of the restricted problem are

$$\ddot{\underline{R}} = -\mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} \quad (1)$$

where  $\underline{R}$  is the position vector of the vehicle in a coordinate system fixed in space,  $\underline{R}_1$  and  $\underline{R}_2$  are respectively the position vectors of the vehicle from earth and moon (with magnitudes  $r_1$  and  $r_2$ ), and  $\mu$  and  $\mu'$  are the gravitational constant times mass of the earth and moon, respectively. Since the barycenter (center of mass of earth and moon) may be regarded as a point fixed in space, the vector  $\underline{R}$  will henceforth be regarded as relative to a system fixed in space with origin at the barycenter. The earth and moon are taken as moving in circles about the barycenter with angular velocity vector  $\underline{\Omega}$ . To use the two fixed center problem as an approximation to the restricted problem, it is necessary to write the equations of motion in a coordinate system in which the earth and moon are fixed. Such a system is one rotating about the barycenter with angular velocity  $\underline{\Omega}$  relative to the fixed system. Denoting the position vector in the rotating system by  $\underline{R}'$ , the equations of motion (1) become

$$\ddot{\underline{R}}' = -\mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} - 2 \underline{\Omega} \times \dot{\underline{R}}' - \underline{\Omega} \times (\underline{\Omega} \times \underline{R}') \quad (2)$$

It is readily shown that the Lagrangian for the equations of motion (2) is

$$\mathcal{L} = \frac{1}{2} \dot{\underline{R}}'^2 + \underline{\Omega} \cdot \underline{R}' \times \dot{\underline{R}}' + \frac{1}{2} (\underline{\Omega} \times \underline{R}')^2 + \frac{\mu}{r_1} + \frac{\mu'}{r_2} \quad (3)$$

and hence the momentum vector conjugate to the position vector  $\underline{R}'$  is given by

$$\underline{P} = \text{grad}_{\underline{R}'} \mathcal{L} = \dot{\underline{R}}' + \underline{\Omega} \times \underline{R} \quad (4)$$

and the Hamiltonian for the problem is

$$H = \underline{P} \cdot \dot{\underline{R}}' - \mathcal{L} = \frac{1}{2} P^2 - \underline{\Omega} \cdot \underline{R}' \times \underline{P} - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \quad (5)$$

and the Hamiltonian equations are

$$\dot{\underline{P}} = -\text{grad}_{\underline{R}'} H = -\underline{\Omega} \times \underline{P} - \mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} \quad (6)$$

and

$$\dot{\underline{R}}' = \text{grad}_{\underline{P}} H = \underline{P} - \underline{\Omega} \times \underline{R}' \quad (7)$$

It will be noted that Eq. (7) is equivalent to Eq. (4), and that if  $\underline{P}$  is replaced using Eq. (4), then Eq. (6) will yield the equations of motion (2).

The solution of the restricted problem will be carried out by making use of a transformation theorem (Reference 1, Chapter 11 and Reference 12, pp 237 to 246) which states that if the Hamiltonian of a system is  $H(q_i, p_i, t)$  with  $q_i$  and  $p_i$  canonically conjugate coordinates so that the Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (8)$$

are satisfied and if  $\psi(q_i, P_i, t)$  is any function, then the variables  $Q_i$  and  $P_i$  defined by

$$Q_i = \frac{\partial \psi}{\partial P_i} = Q_i(q_i, P_i, t), \quad p_i = \frac{\partial \psi}{\partial q_i} = p_i(q_i, P_i, t) \quad (9)$$

are canonical variables for a new Hamiltonian

$$\bar{H} = H + \frac{\partial \psi}{\partial t} \quad (10)$$

regarded as a function of  $Q_i$ ,  $P_i$  and  $t$ , so that

$$\dot{Q}_i = \frac{\partial \bar{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \bar{H}}{\partial Q_i} \quad (11)$$

Now let the Hamiltonian be separated into two terms

$$H = H_1(q_i, p_i) + H_2(q_i, p_i, t) \quad (12)$$

with  $H_1$  independent of the time and such that the partial differential equation

$$H_1(q_i, \frac{\partial \psi_1}{\partial q_i}) + \frac{\partial \psi_1}{\partial t} = 0 \quad (13)$$

possesses a solution for  $\psi_1$ . It is seen that if the function  $\psi_1$  is used in the transformation theorem then the Hamilton equations become

$$\begin{aligned} \dot{Q}_i &= \frac{\partial (H_1 + H_2 + \frac{\partial \psi_1}{\partial t})}{\partial P_i} = \frac{\partial H_2}{\partial P_i} \\ \dot{P}_i &= \frac{\partial (H_1 + H_2 + \frac{\partial \psi_1}{\partial t})}{\partial Q_i} = -\frac{\partial H_2}{\partial Q_i} \end{aligned} \quad (14)$$

by virtue of the defining Eq. (13) for  $\psi_1$ . Further, from Eq. (13), it is evident that, since  $H_1$  is independent of time,

$$\psi_1 = -ht + W(q_i, P_i) \quad (15)$$

with

$$H_1(q_i, \frac{\partial W}{\partial q_i}) - h = 0 \quad (16)$$

and the momenta  $P_i$  must be identified with the constants of integration of Eq. (16) and  $h$ , the separation constant for the time. This is not to be interpreted as meaning that the  $P_i$  are constants of the motion for the Hamiltonian  $H$ . If this were so, the right-hand sides of Eq. (14) would have to vanish. What the solution of Eq. (16) for  $W$  does is to specify a function of  $q_i$  and three new variables  $P_i$ .

This function may be used to invert Eqs. (9) to obtain  $q_i$  and  $p_i$  in terms of the new variables  $P_i$  and three others  $Q_i$ . These expressions for  $q_i$  and  $p_i$  may now be inserted in  $H_2$  for use in Eqs. (14) from which  $Q_i$  and  $P_i$  may now be obtained as functions of time. The solution of the problem associated with  $H$  will then be given by substituting the solutions  $Q_i$  and  $P_i$  of Eqs. (14) in the expression for  $q_i$  and  $p_i$ .

To actually carry out the inversion of Eqs. (9) it must be noted that the functional form of  $\psi_1$  does not depend on the disturbing function ultimately to be used. It depends rather on how the identification of the  $P_i$  is made with the integration constants arising in Eq. (16). The conventional procedure is to regard  $H_1$  as the Hamiltonian of a new problem and identify the  $P_i$  with the action variables  $J_i$  of this new problem. The action variables are always three independent functions of the integration constants and hence are themselves constant for the problem associated with  $H_1$ . Once the functional relation between the  $P_i$  and the integration constants is determined, by identifying the  $P_i$  with the action variables  $J_i$  of  $H_1$ , the conjugate coordinates  $Q_i$  are defined by Eq. (9). It will always happen that  $P_i$  and  $Q_i$  so defined are constant if the Hamiltonian is  $H_1$  because from Eq. (13)

$$\dot{Q}_i = \frac{\partial (H_1 + \frac{\partial \psi_1}{\partial t})}{\partial P_i} = 0 \quad (17)$$

$$J_i = \dot{P}_i = - \frac{\partial (H_1 + \frac{\partial \psi_1}{\partial t})}{\partial Q_i} = 0$$

Once the functional relation between  $q_i$  and  $p_i$  and  $Q_i$  and  $J_i$  is established, however, the problem associated with  $H_1$  is no longer of interest. The disturbing function  $H_2$  is expressed in terms of  $Q_i$  and  $J_i$  and the solution of the problem associated with  $H$  is obtained by integrating Eqs. (14).

A slightly different formulation of the problem is obtained if the time independent function  $W$  of Eq. (15) is used as the generating function of the transformation rather than  $\psi_1$ . The variables  $w_i$  conjugate to the action variables

$J_i$  are the angle variables of the problem associated with  $H_1$ . The relations between the  $w_i$  and the  $Q_i$  are given by

$$w_i = \frac{\partial W}{\partial J_i} = \frac{\partial (\psi_1 + ht)}{\partial J_i} = Q_i + t \frac{\partial h}{\partial J_i} = \nu_i t + Q_i \quad (18)$$

with

$$\nu_i = + \frac{\partial h}{\partial J_i} \quad (19)$$

being functions of the action variables. The perturbation equations for these variables will be given, according to the transformation theorem, by

$$\begin{aligned} \dot{w}_i &= \frac{\partial (H_1 + H_2 + \frac{\partial W}{\partial t})}{\partial J_i} = \frac{\partial H_2}{\partial J_i} + \nu_i \\ \dot{J}_i &= - \frac{\partial (H_1 + H_2 + \frac{\partial W}{\partial t})}{\partial w_i} = - \frac{\partial H_2}{\partial w_i} \end{aligned} \quad (20)$$

since  $W$  is independent of time and  $H_1 = h$  depends only on the action variables and not on the angle variables. The advantage of using the angle variables rather than the  $Q_i$  is that it will always be possible to expand  $H_2$  in a multiple Fourier series in the angle variables and eliminate its explicit dependence on time.

To use the two fixed center problem to solve the restricted problem, the Hamiltonian (5) for the restricted problem may be separated into terms  $H_1$  and  $H_2$  as follows:

$$H_1 = \frac{1}{2} p^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \quad (21)$$

$$H_2 = - \underline{\Omega} \cdot \underline{R}' \times \underline{P} \quad (22)$$

If  $H_1$  is regarded as a Hamiltonian, the associated Hamilton equations are

$$\underline{R}' = - \text{grad}_{\underline{P}} H_1 = \underline{P} \quad (23)$$

and

$$\dot{\underline{P}} = - \text{grad}_{\underline{R}'} H_1 = - \mu \frac{\underline{R}_1}{r_1^3} - \mu' \frac{\underline{R}_2}{r_2^3} = \ddot{\underline{R}}' \quad (24)$$

These last equations are just the equations of motion for the two fixed center problem, so that  $H_1$  is the Hamiltonian of the two fixed center problem. Thus the procedure will be first to find the action and angle variables of the two fixed center problem and then express the disturbing function

$$H_2 = - \underline{\Omega} \cdot \underline{R}' \times \underline{P} \quad (25)$$

in terms of these variables.

Before proceeding with the details of this procedure, it is desirable to make two further transformation of the coordinates. The first will be to a coordinate system with the origin at the midpoint of the earth-moon line with the earth and moon on the x-axis at  $(c,0,0)$  and  $(-c,0,0)$  respectively. The distance between the earth and moon is thus  $2c$ . The z-axis will be taken in the direction of  $\underline{\Omega}$ . The only term in the Hamiltonian affected by this transformation is the  $\underline{\Omega} \cdot \underline{R}' \times \underline{P}$  term in which  $\underline{R}'$  is measured from the barycenter. From Figure I it is evident

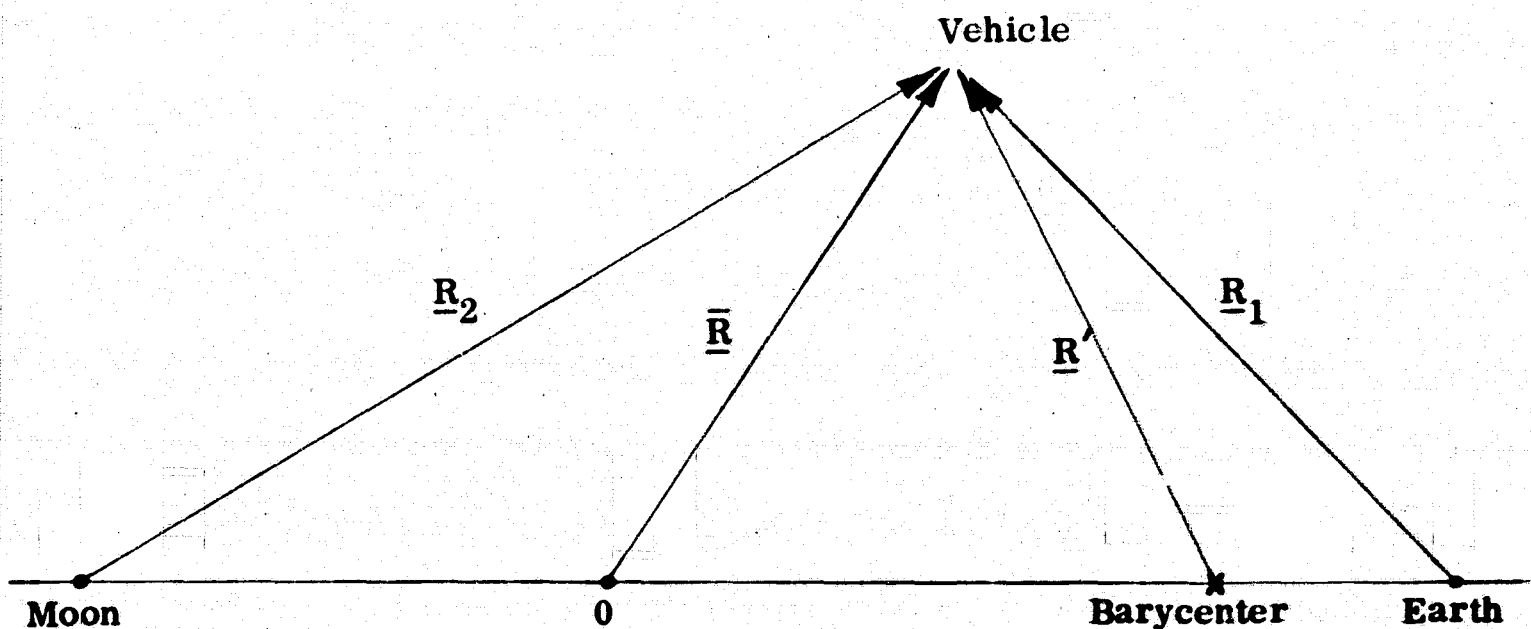


Figure I

that, since the barycenter is at the point  $(c \frac{\mu - \mu'}{\mu + \mu'}, 0, 0)$  the position vectors of the vehicle relative to the midpoint are related by

$$\underline{R}' = \underline{\bar{R}} - i c \frac{\mu - \mu'}{\mu + \mu'} \quad (26)$$

where  $i$  is the unit vector in the  $x$ -direction. Thus the disturbing function becomes

$$H_2 = - \underline{\Omega} \cdot \underline{R}' \cdot \underline{P} = - \underline{\Omega} \cdot \underline{\bar{R}} \times \underline{P} + c \omega \frac{\mu - \mu'}{\mu + \mu'} (j \cdot \underline{P}) \quad (27)$$

where  $j$  is a unit vector in the  $y$  direction.

The second transformation will be from rectangular to prolate ellipsoidal coordinates, in which the Hamilton-Jacobi equation for the two fixed center problem is separable. This transformation may be effected by the generating function

$$F = c q_1 q_2 P_x + c \sqrt{(q_1^2 - 1)(1 - q_2^2)} \cos \varphi P_y + c \sqrt{(q_1^2 - 1)(1 - q_2^2)} \sin \varphi P_z \quad (28)$$

with the new coordinates  $q_1, q_2, \varphi, p_1, p_2$ , and  $p_\varphi$  related to the old ones  $x, y, z, P_x, P_y$ , and  $P_z$  by

$$\begin{aligned} x &= \frac{\partial F}{\partial P_x} & p_1 &= \frac{\partial F}{\partial q_1} \\ y &= \frac{\partial F}{\partial P_y} & p_2 &= \frac{\partial F}{\partial q_2} \\ z &= \frac{\partial F}{\partial P_z} & p_\varphi &= \frac{\partial F}{\partial \varphi} \end{aligned} \quad (29)$$

From the equations for  $x$ ,  $y$ , and  $z$  it is seen that

$$\begin{aligned} x &= c q_1 q_2 \\ y &= c \sqrt{(q_1^2 - 1)(1 - q_2^2)} \cos \varphi \\ z &= c \sqrt{(q_1^2 - 1)(1 - q_2^2)} \sin \varphi \end{aligned} \quad (30)$$

In this system the surfaces  $q_1 = \text{const} \geq 1$  are ellipsoids of revolution about the  $x$ -axis confocal about the earth and moon. The limiting surface  $q_1 = 1$  is the portion of the  $x$ -axis between the earth and moon, and the ellipsoids increase in size with increasing  $q_1$ . The surfaces  $-1 \leq q_2 = \text{const} \leq 1$  are hyperboloids of revolution about the  $x$ -axis, confocal about the earth and moon. The limiting surfaces  $q_1 = 1$  and  $q_2 = -1$  are the portions of the  $x$ -axis to the right of the earth and to the left of the moon, respectively. The surface  $q_2 = 0$  is the  $y$ - $z$  plane and surfaces corresponding to positive values of  $q_2$  are hyperboloids concave towards the earth while those corresponding to negative values of  $q_2$  are concave towards the moon. The angle  $\varphi$  is measured in the  $y$ - $z$  plane about the  $x$ -axis and is zero in the portion of the  $x$ - $y$  plane for which  $y > 0$ . From Eq. (30), it is easy to show that  $r_1$  and  $r_2$  which appear in the Hamiltonian (5) are given by

$$\begin{aligned} r_1 &= c (q_1 - q_2) \\ r_2 &= c (q_1 + q_2) \end{aligned} \quad (31)$$

The equations for  $p_1$ ,  $p_2$ ,  $p_\varphi$  are

$$\begin{aligned} p_1 &= c q_2 P_x + \frac{c q_1 (1 - q_2^2) \cos \varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_y + \frac{c q_1 (1 - q_2^2) \sin \varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_z \\ p_2 &= c q_1 P_x - \frac{c q_2 (q_1^2 - 1) \cos \varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_y - \frac{c q_2 (q_1^2 - 1) \sin \varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} P_z \end{aligned} \quad (32)$$

$$p_{\varphi} = -c \sqrt{(q_1^2 - 1)(1 - q_2^2)} \sin \varphi P_y + c \sqrt{(q_1^2 - 1)(1 - q_2^2)} \cos \varphi P_z \quad (32)$$

Inverting these equations to obtain  $P_x$ ,  $P_y$  and  $P_z$  in terms of  $p_1$ ,  $p_2$  and  $p_{\varphi}$  one obtains for  $H_1$

$$\begin{aligned} H_1 &= \frac{1}{2} P^2 - \frac{\mu}{r_1} - \frac{\mu'}{r_2} \\ &= \frac{1}{2c^2} \left[ \frac{(q_1^2 - 1) p_1^2}{q_1^2 - q_2^2} + \frac{(1 - q_2^2) p_2^2}{q_1^2 - q_2^2} + \frac{p_{\varphi}^2}{(q_1^2 - 1)(1 - q_2^2)} \right] \\ &\quad - \frac{\mu}{c(q_1 - q_2)} - \frac{\mu'}{c(q_1 + q_2)} \end{aligned} \quad (33)$$

and for the disturbing function

$$\begin{aligned} H_2 &= \omega \left\{ \frac{\sqrt{(q_1^2 - 1)(1 - q_2^2)}}{q_1^2 - q_2^2} \cos \varphi \left[ p_1 q_2 - p_2 q_1 + \frac{\mu - \mu'}{\mu + \mu'} (p_1 q_1 - p_2 q_2) \right] \right. \\ &\quad \left. - \frac{p_{\varphi} \sin \varphi}{\sqrt{(q_1^2 - 1)(1 - q_2^2)}} \left( q_1 q_2 + \frac{\mu - \mu'}{\mu + \mu'} \right) \right\} \end{aligned} \quad (34)$$

This completes the preliminary discussion of the problem. The following sections contain the solution of the two fixed center problem which will be useful in the subsequent determination of the generating function  $W$  from Eq. (16) and the action and angle variables for the two fixed center problem which will be the  $w_i$  and  $J_i$  of the perturbation Eqs. (19).

### SECTION III - SOLUTION OF THE TWO FIXED CENTER PROBLEM

The Hamiltonian for the two fixed center problem, obtained in the last section is

$$H = \frac{1}{2c^2} \left\{ \frac{q_1^2 - 1}{q_1^2 - q_2^2} p_1^2 + \frac{(1 - q_2^2)}{q_1^2 - q_2^2} p_2^2 + \frac{p_\varphi^2}{(q_1^2 - 1)(1 - q_2^2)} \right\} - \frac{\mu}{c(q_1 - q_2)} - \frac{\mu'}{c(q_1 + q_2)} \quad (35)$$

The generating function  $W(q_1, q_2, \varphi, P_1, P_2, P_3)$ , which will ultimately be used to obtain the  $w_i$  and  $P_i$  for the perturbation equations is also a very convenient device for obtaining a direct solution to the two fixed center problem. Recalling that for the transformation to be canonical, one must have

$$\begin{aligned} p_1 &= \frac{\partial W}{\partial q_1} \\ p_2 &= \frac{\partial W}{\partial q_2} \\ p_\varphi &= \frac{\partial W}{\partial \varphi} \end{aligned} \quad (36)$$

and

$$Q_i = \frac{\partial W}{\partial P_i} \quad (37)$$

Replacement of  $p_1$ ,  $p_2$  and  $p_\varphi$  by the partials of  $W$  with respect to  $q_1$ ,  $q_2$ , and  $\varphi$ , respectively, in Eq. (35) gives a partial differential equation for  $W$  which is separable. That is, a solution of the form

$$W = W_1(q_1, P_1) + W_2(q_2, P_2) + W_3(\varphi, P_3) \quad (38)$$

exists. It is a fairly simple matter to verify that

$$\begin{aligned}
\left(\frac{dW_1}{dq_1}\right)^2 &= \left(\frac{\partial W}{\partial q_1}\right)^2 = p_1^2 = \frac{2c^2}{(q_1^2 - 1)^2} R^2(q_1) \\
\left(\frac{dW_2}{dq_2}\right)^2 &= \left(\frac{\partial W}{\partial q_2}\right)^2 = p_2^2 = \frac{2c^2}{(1 - q_2^2)^2} S^2(q_2) \\
\left(\frac{dW_3}{d\varphi}\right)^2 &= \left(\frac{\partial W}{\partial \varphi}\right)^2 = p_\varphi^2 = \alpha^2
\end{aligned} \tag{39}$$

where

$$R^2(q_1) = (q_1^2 - 1) \left( h q_1^2 + \frac{\mu + \mu'}{c} q_1 - \beta \right) - \frac{\alpha^2}{2c^2} \tag{40}$$

$$S^2(q_2) = (1 - q_2^2) \left( -h q_2^2 + \frac{\mu - \mu'}{c} q_2 + \beta \right) - \frac{\alpha^2}{2c^2} \tag{41}$$

In Equations (40) and (41),  $h$  is the constant energy of the two fixed center problem and is to be identified with the constant  $h$  of Equation (15) in the previous section. The separation constants are  $\alpha$  and  $\beta$ . It is easily shown that  $\alpha$  is the  $x$ -component of angular momentum about the line of centers. The constant  $\beta$  has no such simple interpretation.

At this stage everything necessary for the solution of the two fixed center problem is available; further discussion of the generating function will be deferred to the next section.

The Hamilton equations for the two fixed center problem give the time derivatives of  $q_1$ ,  $q_2$  and  $\varphi$  as

$$\begin{aligned}
\dot{q}_1 &= \frac{\partial H_1}{\partial p_1} = \frac{p_1}{c^2} \frac{q_1^2 - 1}{q_1^2 - q_2^2} \\
\dot{q}_2 &= \frac{\partial H_1}{\partial p_2} = \frac{p_2}{c^2} \frac{1 - q_2^2}{q_1^2 - q_2^2}
\end{aligned} \tag{42}$$

$$\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{c^2 (q_1^2 - 1) (1 - q_2^2)} \quad (42)$$

Combination of these equations with Equation (39) yields

$$\begin{aligned} q_1 &= \frac{\sqrt{2}}{c} \frac{R(q_1)}{q_1^2 - q_2^2} \\ q_2 &= \frac{\sqrt{2}}{c} \frac{S(q_2)}{q_1^2 - q_2^2} \\ \phi &= \frac{\alpha}{c^2 (q_1^2 - 1) (1 - q_2^2)} \end{aligned} \quad (43)$$

A preferable form for these equations is the following in which a parameter  $u$  is introduced which completes the separation of the variables:

$$\frac{dq_1}{R} = \frac{dq_2}{S} = du \quad (44)$$

$$dt = \frac{c}{\sqrt{2}} (q_1^2 - q_2^2) du \quad (45)$$

$$d\phi = \frac{\alpha}{c\sqrt{2}} \left[ \frac{1}{q_1^2 - 1} + \frac{1}{1 - q_2^2} \right] du \quad (46)$$

From Equation (44), which leads to elliptic integrals of the first kind,  $q_1$  and  $q_2$  turn out to be expressible as elliptic functions of  $u$ . Using these expressions for  $q_1$  and  $q_2$  in Equations (45) and (46), it is then possible to obtain  $t$  and  $\phi$  as functions of  $u$ . The integration of Equations (45) and (46) involves elliptic integrals of the second and third kinds.

The form of solution of Equation (44) depends on the nature of the roots of the quartic expressions  $R^2$  and  $S^2$ . These roots are uniquely determined by the

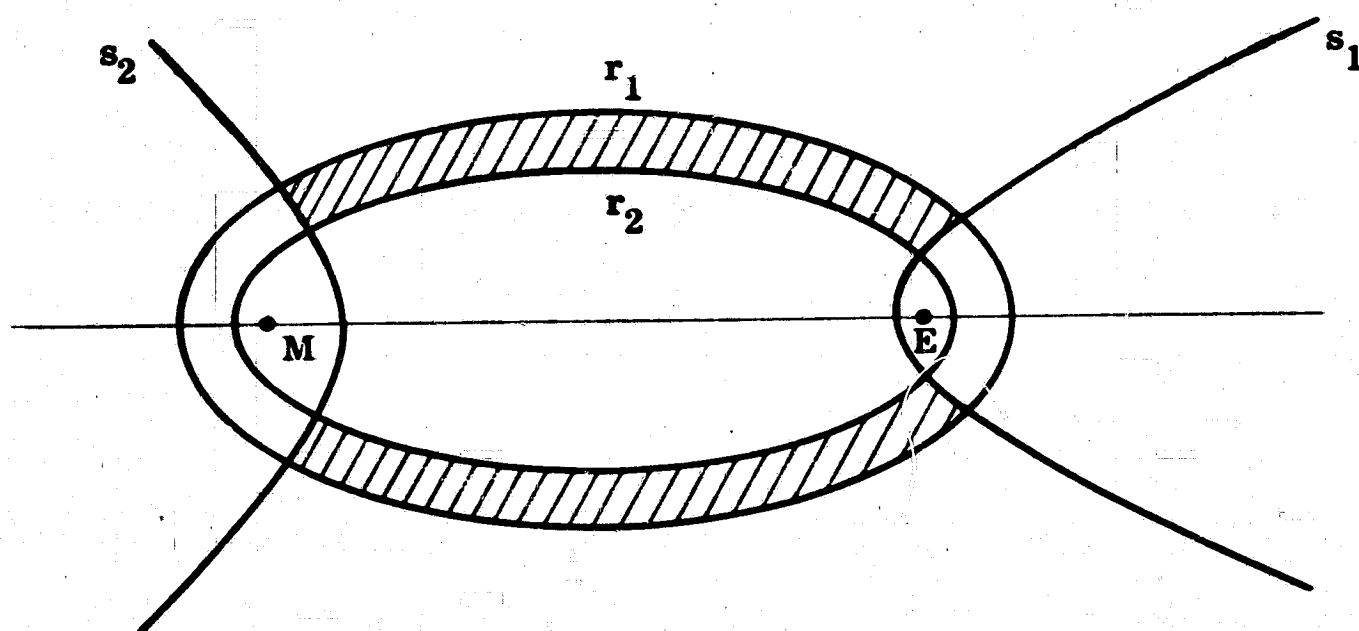
three dynamical constants  $h$ ,  $\alpha$  and  $\beta$ . It is shown in Reference 6 that if  $h < 0$ ,  $R^2$  must have four real roots, two of which exceed unity and the other two lie in the interval  $(\pm 1)$ . Further,  $R^2$  is positive between the largest roots and also between the smallest. Since, however,  $q_1$  must exceed unity, it follows that  $q_1$  is constrained between the largest roots. Thus, if the roots of  $R^2$  in order of decreasing magnitude are denoted by  $r_1, r_2, r_3, r_4$  it may be said that

$$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1 \quad (47)$$

This conclusion may be stated a little differently: the bounds  $r_1$  and  $r_2$  on  $q_1$  represent two ellipsoids (the larger corresponding to  $r_1$ ) which bound the region in space in which the vehicle may move.

The corresponding results for the quartic  $S^2$  are more complicated: none of the roots exceed unity and at least two lie in the interval  $(\pm 1)$ . The other two may also lie in this interval, may be real and both less than  $-1$ , or may be complex. The quartic is positive between the two largest roots and between the two smallest, if they are real. Since  $q_2$  must lie in the interval  $(\pm 1)$  it follows that the orbit is constrained between the two largest roots or between the two smallest if they also lie in the  $(\pm 1)$  interval. If all four roots of  $S^2$  are in  $(\pm 1)$ , knowledge of the position of one point of the orbit specifies whether  $q_2$  is constrained between the largest or the smallest roots; transitions from one band to the other cannot occur, since if  $S^2$  becomes negative,  $q_2$  becomes imaginary. The roots of  $S^2$  in the interval  $(\pm 1)$  correspond to hyperboloids bounding the motion in space.

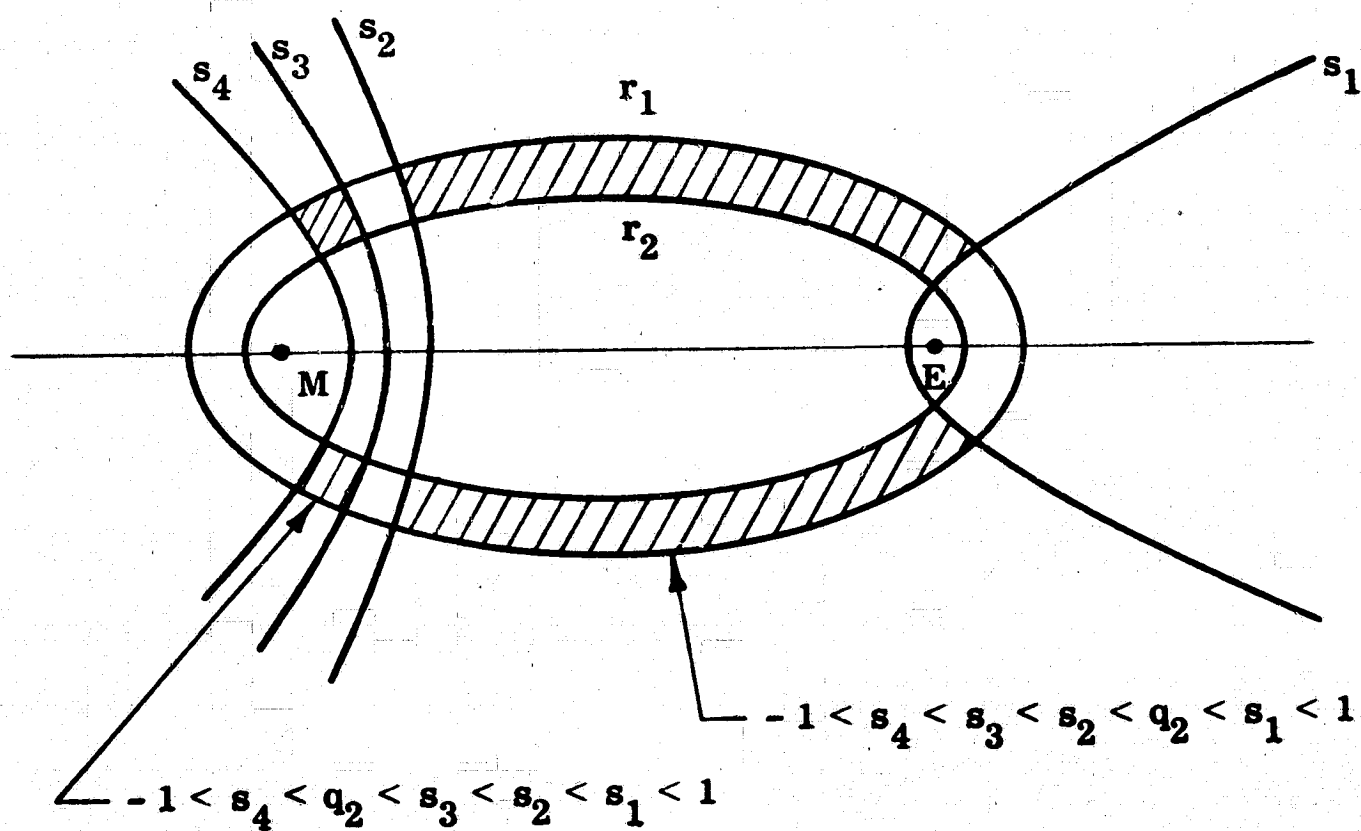
Summarizing the above results for negative energy, two possibilities for bounds on the orbit occur. These are shown in Figures II and III where the shaded areas are regions in which motion may occur.



$$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1$$

$$-1 < s_2 < q_2 < s_1 < 1 \left\{ \begin{array}{l} \text{either } s_3, s_4 < -1 \\ \text{or } s_3, s_4 \text{ complex} \end{array} \right.$$

Figure II



$$-1 < s_4 < s_3 < s_2 < q_2 < s_1 < 1$$

$$-1 < s_4 < q_2 < s_3 < s_2 < s_1 < 1$$

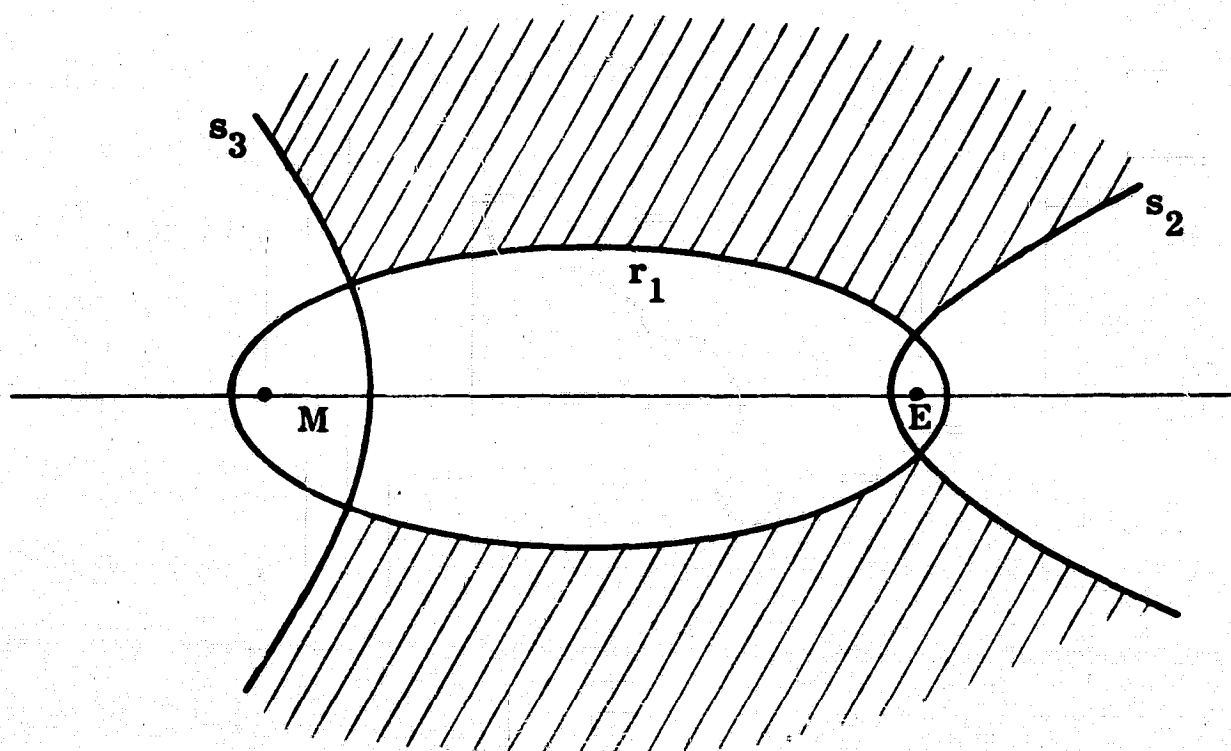
$$-1 < r_4 < r_3 < 1 < r_2 < q_1 < r_1$$

Figure III

If one thinks of  $h$ ,  $\alpha$  and  $\beta$ , which determine all the roots of  $R^2$  and  $S^2$  as being three dynamical specifications of a two fixed center orbit, it is clear that any remaining specifications must not violate the bounds on the region in which the motion can occur. That is, these bounds impose constraints on any further specifications. Actually, not even  $h$ ,  $\alpha$  and  $\beta$  can be arbitrarily selected: they must lead to roots of  $R^2$  satisfying Equation (47) and roots of  $S^2$  satisfying one or the other of the following:

$$\begin{aligned} \text{(a)} \quad & -1 \leq s_2 \leq s_1 \leq 1 \text{ and either } s_3, s_4 < 1 \text{ or } s_3, s_4 \text{ complex} \\ \text{(b)} \quad & -1 \leq s_4 \leq s_3 \leq s_2 \leq s_1 \leq 1 \end{aligned} \quad (48)$$

If the energy is positive, it may be shown that  $R^2$  has one root, say  $r_1$  exceeding unity and is positive for  $q_1$  exceeding  $r_1$ . The other roots are all less than 1. The quartic  $S^2$  has two roots  $s_3 < s_2$  in the interval  $(\pm 1)$ , and one on each side of this interval. It is positive for  $s_3 < q_2 < s_2$ . Thus in this case the motion must take place in the unbounded region shown in Figure IV.



$$\begin{aligned} q &> r_1 > 1 \\ -1 &< s_3 < q_2 < s_2 < 1 \end{aligned}$$

Figure IV

As noted above,  $q_1$  and  $q_2$  are expressible in terms of elliptic functions of  $u$ . The particular elliptic function occurring depends on the nature of the roots. In all cases, see Reference 6,

$$q_i = \frac{A_i f(\alpha_i(u + \beta_i)) + B_i}{C_i f(\alpha_i(u + \beta_i)) - 1} \quad (49)$$

The  $A_i$ ,  $B_i$ ,  $\alpha_i$  are constants depending only on the roots and hence on  $h$ ,  $\alpha$  and  $\beta$ . The constants  $\beta_i$  depend on  $h$ ,  $\alpha$  and  $\beta$  as well as whatever additional specifications are made to select a particular orbit. For  $q_1$ , the function  $f$  is an sn or dn function according as  $h$  is negative or positive. For  $q_2$ ,  $h < 0$ ,  $f$  is an sn or cn function according as all four or only two of the roots are real and if  $h > 0$ ,  $f$  is a dn function. It is evident, of course, that  $q_1$  and  $q_2$  are individually periodic in the variable  $u$ . The periods of  $q_1$  and  $q_2$  are, however, in general non-commensurable, so that the motion in space of the vehicle will, in general, be nonperiodic. The quarter periods of  $q_1$  and  $q_2$  are usually denoted by  $K_1$  and  $K_2$ , respectively, and it may be shown that these quarter periods depend only on the roots of  $R^2$  and  $S^2$ , respectively, and hence only on  $h$ ,  $\alpha$  and  $\beta$ . From the way in which the  $\beta_i$  occur in Equation (49), it is evident that they represent a phase. In fact, it is assumed in Equation (49) that  $u = 0$  corresponds to some point on the orbit, say the initial point, and the  $\beta_i$  represent the variation in  $u$  required to get from this point to one of the extreme values of  $q_i$  - that is, to a point of tangency with one of the bounding ellipsoids for  $q_1$ , and with one of the bounding hyperboloids for  $q_2$ .

The integration of the equations for time and  $\varphi$  leads in all cases to the following forms (consult Reference 6)

$$t = (n_1 - n_2)u + F_1(u) + F_2(u) \quad (50)$$

$$\varphi = (m_1 + m_2)u + G_1(u) + G_2(u) \quad (51)$$

where  $n_1$  and  $m_1$  are constants depending on the roots of  $R^2$ , and  $n_2$  and  $m_2$  depend

on the roots of  $S^2$ . For negative  $h$ , the functions  $F_1(u)$  and  $G_1(u)$  are periodic functions of  $u$  with period  $2K_1$ , while  $F_2(u)$  and  $G_2(u)$  are periodic with period  $2K_2$ . For positive  $h$ , the functions  $F_i$  and  $G_i$  become logarithmic.

#### SECTION IV. DETERMINATION OF THE GENERATING FUNCTION

The differential equations for the generating function, Eqs. (39), may be written

$$\begin{aligned} \frac{dW_1}{dq_1} &= \frac{\partial W}{\partial q_1} = \frac{\sqrt{2} c}{q_1^2 - 1} & R \\ \frac{dW_2}{dq_2} &= \frac{\partial W}{\partial q_2} = \frac{\sqrt{2} c}{1 - q_2^2} & S \\ \frac{dW_3}{d\varphi} &= \frac{\partial W}{\partial \varphi} = \alpha \end{aligned} \quad (52)$$

These are ordinary differential equations, and integration again leads to elliptic integrals. Before carrying out the integration, however, some discussion of the limits on the integrals is necessary. It will be recalled that the generating function was to be a function of six variables.

$$W = W(q_1, q_2, \varphi, P_1, P_2, P_3) \quad (53)$$

and the differential equations (52) give only three of the six partial derivatives of  $W$ . Now the dependence of  $W$  on  $q_1$ ,  $q_2$  and  $\varphi$  can be carried by the upper limits of the integrals resulting from Eqs. (52). These upper limits should be simply  $q_1$ ,  $q_2$ , and  $\varphi$ , respectively. Recalling further that the momenta  $P_i$  are supposed to be constants, and noting that three independent constants  $h$ ,  $\alpha$  and  $\beta$  already are explicitly in Eq. (52), it is evident that these three constants or some three independent functions of them must be identified with  $P_i$ . It is convenient at present to identify  $h$ ,  $\alpha$  and  $\beta$  themselves with  $P_i$  and defer to a later stage in the development any more complicated identification. If this is done, it now becomes obvious that the lower limits on the integrals must be either functions of  $h$ ,  $\alpha$  and  $\beta$  or absolute constants. This is so first because  $W$  is a function only of  $q_1$ ,  $q_2$ ,  $\varphi$

and the  $P_i$ , and, since the integrals will be functions of their limits, only these quantities and absolute constants may be included in the limits. Secondly, the upper limits have already been taken as  $q_1$ ,  $q_2$ , and  $\varphi$ , and recalling that the partials of  $W$  with respect to  $q_1$ ,  $q_2$  and  $\varphi$  must be  $p_1$ ,  $p_2$  and  $p_\varphi$ , no further dependence of  $W$  on  $q_1$ ,  $q_2$  and  $\varphi$  can be allowed without modifying the  $p$ 's from which the equations (52) for  $W$  were obtained in the first place. The only remaining problem, then, is to select lower limits which depend only on  $h$ ,  $\alpha$  and  $\beta$ . For the integral for  $W_1$ , the variable is  $q_1$  which has bounds on its variation. The bounds depend on  $h$ ,  $\alpha$  and  $\beta$ , and since  $r_1$  is a bound whether the energy is positive or negative, it is a satisfactory lower limit. For  $W_2$  the bounds vary with the particular conditions of the problem. However, for orbits approaching both Earth and Moon, the bound  $s_2$  always occurs, and will be selected as the lower limit. For  $W_3$ , the situation is a little different. The variable is  $\varphi$ , and reference to Eq. (43) shows that  $\varphi$  has the sign of  $\alpha$  and is thus monotone. Hence any absolute constant is acceptable as a lower limit and 0 will be selected. The generating function may now be written:

$$W(q_1, q_2, \varphi, h, \alpha, \beta) = W_1(q_1, h, \alpha, \beta) + W_2(q_2, h, \alpha, \beta) + W_3(\varphi, h, \alpha, \beta)$$

$$= \sqrt{2} c \int_{r_1}^{q_1} \frac{R}{q_1^2 - 1} dq_1 + \sqrt{2} c \int_{s_2}^{q_2} \frac{S}{1 - q_2^2} dq_2 + \alpha \varphi \quad (54)$$

where  $W_3$  is integrable directly. It might be remarked at this stage that there is an essential difference between this generating function and the corresponding function for the Kepler problem. The upper limits in the integral occurring in both generating functions may be regarded as the coordinates of a point on the orbit. In the Kepler problem, the lower limits correspond to the perigee distance for the radial integral and to zero for the two angle integrals. This may be regarded as a point on any orbit, since the angles may just be measured from the perigee point. In the two fixed center problem however, the lower limits  $r_1$ ,  $s_2$  and 0 may be regarded as a point only on a very special orbit -- namely, one which is simultaneously tangent to the ellipsoid  $r_1$  and the hyperboloid  $s_2$ , and this tangency must occur in the  $x$ - $y$  plane.

To complete the canonical transformation generated by  $W$ , the  $P_i$  will be identified with  $h$ ,  $\alpha$  and  $\beta$  as follows:

$$\begin{aligned} P_1 &= P_h = h \\ P_2 &= P_\beta = \beta \\ P_3 &= P_\alpha = \alpha \end{aligned} \tag{55}$$

The conjugate coordinates  $Q_i$  then become

$$\begin{aligned} Q_1 &= Q_h = \frac{\partial W}{\partial h} = \frac{c}{\sqrt{2}} \int_{r_1}^{q_1} \frac{q_1^2 dq_1}{R} - \frac{c}{\sqrt{2}} \int_{s_2}^{q_2} \frac{q_2^2 dq_2}{S} \\ Q_2 &= Q_\beta = \frac{\partial W}{\partial \beta} = -\frac{c}{\sqrt{2}} \int_{r_1}^{q_1} \frac{dq_1}{R} + \frac{c}{\sqrt{2}} \int_{s_2}^{q_2} \frac{dq_2}{S} \\ Q_3 &= Q_\alpha = \frac{\partial W}{\partial \alpha} = -\frac{\sqrt{2} \alpha}{c} \int_{r_1}^{q_1} \frac{dq_1}{(q_1^2 - 1) R} - \frac{\sqrt{2} \alpha}{c} \int_{s_2}^{q_2} \frac{dq_2}{(1 - q_2^2) S} + \varphi \end{aligned} \tag{56}$$

In differentiating the integrals in  $W$  there are really three terms for each integral: one is the integral of the derivative of the integrand and the other two are obtained by evaluating the integrand at the limits and multiplying by the derivatives of the limits. The terms corresponding to the limits vanish, because the upper limits are not functions of  $h$ ,  $\alpha$  and  $\beta$ , the integrands for the  $q_1$  and  $q_2$  integrals vanish at the lower limits, and the lower limit of the  $\varphi$  integral is an absolute constant.

It will be noted that all the integrals occurring in Eq. (56) have forms identical with one or another of those occurring in Eqs. (44), (45) and (46) for the determination of  $q_1$ ,  $q_2$ ,  $t$  and  $\varphi$  as functions of  $u$ . The only difference is that in reference 6, where the integration of Eqs. (44), (45) and (46) is carried out in all detail, the lower limit on  $u$  was taken as zero. Here the lower limits are roots of  $R^2$  and  $S^2$ .

Of the three  $Q_i$ ,  $Q_\beta$  has a relatively simple interpretation if one replaces  $dq_1$  and  $dq_2$  by  $du$  in accordance with Eq. (44). Then  $Q_\beta$  becomes

$$Q_\beta = - \left[ \frac{c}{\sqrt{2}} \int_{u(r_1)}^{u(q_1)} du - \int_{u(s_2)}^{u(q_2)} du \right] \quad (57)$$

$$= \frac{c}{\sqrt{2}} (u(r_1) - u(s_2))$$

since the upper limits correspond to a point on the orbit and therefore represent the same value of  $u$ . Thus  $Q_\beta$  appears proportional to the variation in  $u$  associated with a transit from tangency with a hyperboloid to tangency with an ellipsoid. Since the orbit is not, in general, periodic this statement does not yet uniquely define  $Q_\beta$ . To arrive at such a definition, it may be noted that in terms of the canonical variables  $P_i$  and  $Q_i$  the Hamiltonian becomes

$$H = h = P_1 \quad (58)$$

so that the Hamilton equations in these variables are:

$$\dot{P}_1 = \dot{P}_2 = \dot{P}_3 = \dot{h} = \dot{\alpha} = \dot{\beta} = 0 \quad (59)$$

and

$$\dot{Q}_\alpha = \dot{Q}_\beta = 0, \quad \dot{Q}_h = 1 \quad (60)$$

therefore

$Q_\alpha$  and  $Q_\beta$  are constants and

$$Q_h = t + \text{const} = t + C \quad (61)$$

The values of  $h$ ,  $\alpha$  and  $\beta$  may be obtained from a set of initial conditions. The values of  $Q_\alpha$ ,  $Q_\beta$  and  $C$  may be obtained from the initial conditions also, provided it is agreed that the  $q_1 = r_1$  and  $q_2 = s_2$  are to be associated, say, with the tangencies to the ellipsoid  $r_1$  and the hyperboloid  $s_2$  closest to the initial point. Other identifications of  $q_1 = r_1$  and  $q_2 = s_2$  will lead to  $Q$ 's differing from those just defined by multiples of the periods  $K_1$  and  $K_2$ .

If one applies the same analysis to  $Q_h$  and  $Q_\alpha$  as used for  $Q_\beta$  (replacing  $dq_1$  and  $dq_2$  by  $u$ ), the following expressions are obtained:

$$Q_h = t + \frac{c}{\sqrt{2}} \left[ \int_{u(r_1)}^0 q_1^2 du - \int_{u(s_2)}^0 q_2^2 du \right] \quad (62)$$

or

$$C = \frac{c}{\sqrt{2}} \left[ \int_{u(r_1)}^0 q_1^2 du - \int_{u(s_2)}^0 q_2^2 du \right] \quad (63)$$

and

$$Q_\alpha = -\frac{\sqrt{2}\alpha}{c} \left[ \int_{u(r_1)}^0 \frac{du}{q_1^2 - 1} - \int_{u(s_2)}^0 \frac{du}{1 - q_2^2} \right] \quad (64)$$

## SECTION V. ACTION AND ANGLE VARIABLES

The action and angle variables are conventionally defined only for conditionally periodic systems, which means that for the two fixed center problem, the development can be made only for  $h < 0$ . The action variables are defined in terms of the generating function  $W$ , as follows:

$$\begin{aligned}
 J_1 &= \oint \frac{\partial W}{\partial q_1} dq_1 = \sqrt{2} c \oint \frac{R dq_1}{q_1^2 - 1} \\
 J_2 &= \oint \frac{\partial W}{\partial q_2} dq_2 = \sqrt{2} c \oint \frac{S dq_2}{1 - q_2^2} \\
 J_3 &= \int_0^{2\pi} \frac{\partial W}{\partial \varphi} d\varphi = 2\pi \alpha
 \end{aligned} \tag{65}$$

where the integral for  $J_1$  is taken over a complete cycle of variation of  $q_1$  - i.e. from  $r_1$  to  $r_2$  and back to  $r_1$ , while that for  $J_2$  is over a complete cycle of  $J_2$  from  $s_1$  to  $s_2$  and back to  $s_1$ . These integrals can, for the most part, be reduced to the forms already encountered as follows:

$$\begin{aligned}
 J_1 &= \sqrt{2} c \oint \frac{R dq_1}{q_1^2 - 1} = \sqrt{2} c \oint \frac{R^2}{q_1^2 - 1} \frac{dq_1}{R} \\
 &= \sqrt{2} c \int_0^{4K_1} \left[ h q^2 + \frac{\mu + \mu'}{c} q_1 - \beta - \frac{\tilde{c}^2}{2c^2 (q_1^2 - 1)} \right] du
 \end{aligned} \tag{66}$$

A complete cycle of variation  $q_1$  corresponds to a variation in  $u$  of  $4 K_1$ . Now the first term in this integral has the form of the dependence of the time on  $q_1$ , and, referring to Eq. (50) it is seen that the periodic part  $F_1$  will vanish and hence the contribution of the first term to the integral is  $8 h n_1 K_1$ . Similarly the last term has the form of the  $q_1$  part of the  $\phi$  integral, Eq. (51), and will contribute  $-\alpha \cdot 4 m_1 K_1$ . The  $\beta$  term contributes just  $-\sqrt{2} c \beta \cdot 4 K_1$ . The only new integral to evaluate is

$$\int_0^{4K_1} q_1 \, du \quad (67)$$

This integral, too, turns out to be expressible as a linear term in  $u$  plus a periodic one, so that for the limits given, it contributes a term  $\sqrt{2} (\mu + \mu') \ell_1 \cdot 4 K_1$  where  $\ell_1$  is the coefficient of the linear term. Thus, finally,

$$J_1 = 8 h n_1 K_1 + 4 \sqrt{2} (\mu + \mu') \ell_1 K_1 - 4 \sqrt{2} c \beta K_1 - 4 \alpha m_1 K_1 \quad (68)$$

In an exactly similar fashion

$$J_2 = -8 h n_2 K_2 + 4 \sqrt{2} (\mu - \mu') \ell_2 K_2 + 4 \sqrt{2} c \beta K_2 - 4 \alpha m_2 K_2 \quad (69)$$

To obtain the angle variables conjugate to the action variables, it is necessary to recall that the original condition imposed on the  $P_i$  was only that they be constants. Identification of the  $P_i$  with  $h$ ,  $\alpha$ , and  $\beta$  is only one possibility; any three independent functions of  $h$ ,  $\alpha$ , and  $\beta$  would serve as well and, in particular, it is now desirable to identify  $P_i$  with  $J_i$ . Now the generating function  $W$  is given in Eq. (54) in terms of  $q_1$ ,  $q_2$ ,  $\phi$ ,  $h$ ,  $\alpha$ ,  $\beta$ , and  $r_1$  and  $s_2$ . The roots  $r_1$  and  $s_2$  are, however, functions of  $h$ ,  $\alpha$  and  $\beta$ . Now if Eqs. (68) and (69) together with the third of Eqs. (65) be inverted to express  $h$ ,  $\alpha$ , and  $\beta$  in terms of  $J_1$ ,  $J_2$ , and  $J_3$ , it will be possible to substitute for  $h$ ,  $\alpha$ , and  $\beta$  in  $W$  to obtain  $W$  as a function of  $q_1$ ,  $q_2$ ,  $\phi$ ,  $J_1$ ,  $J_2$ , and  $J_3$ . It should be remarked that the inversion to obtain  $h$ ,  $\alpha$ , and  $\beta$

in terms of  $J_1$ ,  $J_2$  and  $J_3$  is not an easy task since the coefficients  $n_1$ ,  $n_2$ ,  $l_1$ ,  $l_2$ ,  $m_1$ ,  $m_2$  are very complicated functions of  $h$ ,  $\alpha$  and  $\beta$ . Nevertheless the procedure is possible in principle and the angle variables  $w_i$  conjugate to the  $J$ 's are given by the partial derivatives of the generating function  $W$  with respect to the  $J$ 's:

$$w_i = \frac{\partial W}{\partial J_i} \quad (70)$$

One may obtain expressions for the  $w_i$  without actually performing the inversion, by writing the derivatives of  $W$  with respect to  $J_i$  in terms of its derivatives with respect to  $h$ ,  $\alpha$ , and  $\beta$ :

$$\begin{aligned} w_i = \frac{\partial W}{\partial J_i} &= \frac{\partial W}{\partial h} \frac{\partial h}{\partial J_i} + \frac{\partial W}{\partial \alpha} \frac{\partial \alpha}{\partial J_i} + \frac{\partial W}{\partial \beta} \frac{\partial \beta}{\partial J_i} \\ &= Q_h \frac{\partial h}{\partial J_i} + Q_\alpha \frac{\partial \alpha}{\partial J_i} + Q_\beta \frac{\partial \beta}{\partial J_i} \end{aligned} \quad (71)$$

from Eqs. (56) defining the variables conjugate to  $h$ ,  $\alpha$  and  $\beta$ . Or, recalling Eq. (62) for  $Q_h$ ,

$$w_i = (t + C) \frac{\partial h}{\partial J_i} + Q_\beta \frac{\partial \beta}{\partial J_i} + Q_\alpha \frac{\partial \alpha}{\partial J_i} \quad (72)$$

where  $C$ ,  $Q_\alpha$  and  $Q_\beta$  are constants.

The derivatives of  $h$ ,  $\alpha$  and  $\beta$  may be expressed in terms of the  $n$ 's,  $m$ 's,  $l$ 's and  $K$ 's occurring in Eqs. (68) and (69) by first obtaining the partials of the  $J$ 's with respect to  $h$ ,  $\alpha$  and  $\beta$  from Eqs. (65), and then inverting their Jacobian matrix. The results of this calculation for the Jacobian are

$$J \begin{pmatrix} J_1 & J_2 & J_3 \\ h & \beta & \alpha \end{pmatrix} = \begin{bmatrix} 4 n_1 K_1 & -2\sqrt{2} c K_1 & -4 m_1 K_1 \\ -4 n_2 K_2 & 2\sqrt{2} c K_2 & -4 m_2 K_2 \\ 0 & 0 & 2\pi \end{bmatrix} \quad (73)$$

and its inverse is

$$J \begin{pmatrix} h & \beta & \alpha \\ J_1 & J_2 & J_3 \end{pmatrix} = \begin{bmatrix} \frac{1}{4K_1(n_1 - n_2)} & \frac{1}{4K_2(n_1 - n_2)} & \frac{m_1 + m_2}{2\pi(n_1 - n_2)} \\ \frac{\sqrt{2}n_2}{4cK_1(n_1 - n_2)} & \frac{\sqrt{2}n_1}{4cK_2(n_1 - n_2)} & \frac{\sqrt{2}(n_1m_2 - m_1n_2)}{2\pi c(n_1 - n_2)} \\ 0 & 0 & \frac{1}{2\pi} \end{bmatrix} \quad (74)$$

so that, finally

$$\begin{aligned} w_1 &= \frac{t + C}{4K_1(n_1 - n_2)} + \frac{\sqrt{2}n_2 Q_\beta}{4cK_1(n_1 - n_2)} \\ w_2 &= \frac{t + C}{4K_2(n_1 - n_2)} + \frac{\sqrt{2}n_1 Q_\beta}{4cK_2(n_1 - n_2)} \\ w_3 &= \frac{(t + C)(m_1 + m_2)}{2\pi(n_1 - n_2)} + \frac{\sqrt{2}Q_\beta(n_1m_2 + m_1n_2)}{2\pi c(n_1 - n_2)} + \frac{Q_\alpha}{2\pi} \end{aligned} \quad (75)$$

are the angle variables.

## SECTION VI. CONCLUSION

To complete the solution of the restricted problem, it is now necessary to express the disturbing function  $H_2$  in terms of the action and angle variables. This is a formidable problem. The disturbing function is given in terms of  $q_1$ ,  $q_2$ ,  $\varphi$  and their conjugate momenta in Eq. (34). The momenta are given in terms of  $q_1$ ,  $q_2$ ,  $\varphi$ ,  $h$ ,  $\alpha$  and  $\beta$  by Eqs. (39) so that  $H_2$  may readily be written in terms of these variables. Starting from the other end, the action variables  $J_1$  and  $J_2$  are given in terms of complicated functions of  $h$ ,  $\alpha$ , and  $\beta$  [ Eqs. (68) and (69) ] while  $J_3$  is just  $2\pi\alpha$  [ Eqs. (65) ]. The angle variables  $w_i$  are given by Eq. (75) as linear functions of  $Q_h$ ,  $Q_\alpha$ , and  $Q_\beta$  with coefficients which are functions of  $h$ ,  $\alpha$ , and  $\beta$  similar to those occurring for  $J_i$ . And  $Q_h$ ,  $Q_\alpha$ , and  $Q_\beta$  are related to  $q_1$ ,  $q_2$ ,  $\varphi$ , and  $h$ ,  $\alpha$ , and  $\beta$  by Eqs. (56). Thus, the following procedure would yield the information necessary to write  $H_2(w_i, J_i)$ :

1. Express  $K_1$ ,  $K_2$ ,  $\ell_1$ ,  $\ell_2$ ,  $n_1$ ,  $n_2$ ,  $m_1$ ,  $m_2$  as functions of  $h$ ,  $\alpha$ , and  $\beta$ .

2. 
$$\alpha = \frac{J_3}{2\pi}$$

Invert Eqs. (68) and (69) using the results of step 1 to obtain  $h(J_i)$  and  $\beta(J_i)$ .

3. Express  $K_1$ ,  $K_2$ ,  $\ell_1$ ,  $\ell_2$ ,  $n_1$ ,  $n_2$ ,  $m_1$ ,  $m_2$  which are functions of  $h$ ,  $\alpha$  and  $\beta$  in terms of  $J_i$ .

4. Invert Eqs. (75) to obtain  $Q_h = t + c$ ,  $Q_\alpha$  and  $Q_\beta$  as functions of the angle variables  $w_i$  and  $K_1$ ,  $K_2$ ,  $\ell_1$ ,  $\ell_2$ ,  $n_1$ ,  $n_2$ ,  $m_1$ , and  $m_2$ .

5. Use step 1 to obtain  $Q_h$ ,  $Q_\alpha$ , and  $Q_\beta$  as functions of  $w_i$  and  $J_i$ .

6. Invert Eqs. (56) to obtain  $q_1$ ,  $q_2$ , and  $\varphi$  as functions of  $Q_h$ ,  $Q_\alpha$ ,  $Q_\beta$ ,  $h$ ,  $\alpha$ , and  $\beta$ .
7. In the expressions for  $q_1$ ,  $q_2$ , and  $\varphi$  obtained in step 6 replace  $Q_h$ ,  $Q_\alpha$ , and  $Q_\beta$  using step 5 and  $h$ ,  $\alpha$ , and  $\beta$  using step 2 to obtain  $q_1$ ,  $q_2$ , and  $\varphi$  in terms of  $w_i$  and  $J_i$ .
8. In the disturbing function  $H_2(q_1, q_2, \varphi, h, \alpha, \beta)$ , replace  $q_1$ ,  $q_2$ , and  $\varphi$  from step 7 and  $h$ ,  $\alpha$ , and  $\beta$  from step 2 to obtain, finally,  $H_2(w_i, J_i)$ .

Steps 1, 2, and 6 are the difficult ones in this procedure. It is relatively easy to write  $K_1$ ,  $K_2$ ,  $\ell_1$ ,  $\ell_2$ ,  $n_1$ ,  $n_2$ ,  $m_1$ , and  $m_2$  as functions of the roots of the quartics and two intermediate parameters which are related to the roots of the quartics by transcendental equations. The roots of the quartics are, of course, functions of  $h$ ,  $\alpha$ , and  $\beta$ , but it is not easy to write out these functions explicitly. Thus, even step 1 is quite difficult, and to perform the inversion required in step 2 in closed form appears nearly impossible.

It should be remarked, however, that, at least for certain types of orbits, it should be possible to get fairly good approximations of these steps. For a lunar orbit which starts from the earth, closely circles the moon and returns to the earth, it may be shown that  $\alpha^2/2c^2$  is very small. This is so because such an orbit has very close approaches to the line of centers, and recalling that  $\alpha$  is the angular momentum about the line of centers, it follows that  $\alpha$  must be small. If  $\alpha$  were zero, two of the roots of the quartics would be  $\pm 1$  and the other two are obtained in terms of  $h$  and  $\beta$  by solving quadratics [see Eqs. (40) and (41)]. Now it is possible to obtain the roots of the quartics for small  $\alpha$  in terms of those for zero  $\alpha$  in a series of powers of  $\alpha$ . Thus for small  $\alpha$ , it is easy to obtain fairly simple approximate expressions for the roots in terms of  $h$ ,  $\alpha$ , and  $\beta$ . Further, it turns out that the transcendental equations to be inverted for the intermediate parameters are very well approximated by just two terms of an expansion. Thus, it is feasible, for lunar orbits, to obtain a good approximation to steps 1 and 2.

### The complete elliptic integrals

$$\oint q_1^2 du, \quad \oint \frac{dq_1}{q_1^2 - 1}, \quad \oint dq_1$$

and similar ones for  $q_2$ , have forms very similar to those obtained by Vinti<sup>(8)</sup> in his model for the oblate earth. Vinti used oblate spheroidal coordinates for his model and the close connection between his development and that given in this report for the two fixed center problem was first pointed out by Pines<sup>(9)</sup>. The Vinti integrals have recently been evaluated approximately by Izsak<sup>(10)</sup> using a technique developed by Sommerfeld<sup>(11, 12)</sup> for evaluating certain contour integrals of functions with branch points. The method is to expand the integrals in terms of a quadratic function and evaluate the series of resulting integrals about contours enclosing the roots of the quadratic. The values of the integrals so obtained are explicitly in terms of the coefficients of the quartics. For the method to be valid, the expansion must converge over both the original and the final contours. This condition is satisfied for Izsak's expansion of the Vinti integrals. However, none of the obvious expansions for the two fixed center integrals converge over the final contour.

The greatest difficulty in following the procedure for obtaining  $H_2$  is in step 6. Eqs. (56) relating  $Q_h$ ,  $Q_\alpha$ , and  $Q_\beta$  with  $q_1$ ,  $q_2$ , and  $\varphi$  are transcendental equations and it is hard to say how well their inversion could be approximated by some approximation procedure, such as the Lagrange inversion theorem.

It should be remarked that it would be possible to write  $H_2$  in terms of  $Q_h$ ,  $Q_\alpha$ ,  $Q_\beta$ ,  $h$ ,  $\alpha$ , and  $\beta$  rather than in terms of  $w_i$  and  $J_i$ . This is not done in the Kepler problem because the relation between the original coordinates and time is best achieved by a Fourier expansion in the mean anomaly rather than in time. An expansion in time would involve far more complicated coefficients. Which set of variables will turn out to be better for the two fixed center problem is hard to predict at this stage.

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**A Report on Investigations of  
THE RESTRICTED THREE BODY PROBLEM**

1. **The Surfaces of Zero Velocity  
in Regularized Coordinates**
2. **The Euler Problem by  
Graphical Analysis**

by

**W. S. Krogdahl, T. J. Pignani,  
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18695  
**SUMMARY**

Representative zero-velocity curves for the restricted three-body problem have been calculated in bipolar and closely related regularized Thiele coordinates. This development has been the primary interest of Dr. Krogdahl. Dr. Wells and Dr. Pignani have devoted most of their attention to a study of the Euler problem through a graphical analysis as a preliminary approach to the study of this problem by the use of the Weierstrass P-Functions.

## THE SURFACES OF ZERO VELOCITY IN REGULARIZED COORDINATES

The equations of the surfaces of zero velocity in the restricted three-body problem are given by the equation

$$(1) \quad (1 - \mu) \left[ \rho^2 + \frac{2}{\rho} \right] + \mu \left[ \sigma^2 + \frac{2}{\sigma} \right] = C$$

in a system of bipolar coordinates.<sup>1, 2\*</sup> Here  $\rho$  is the distance of an infinitesimal mass from a major mass,  $(1 - \mu)$ , and  $\sigma$  is its distance from the second major mass,  $\mu$  (see Figure 1). The constant  $C$  is the parameter of the family of curves and has the range  $(3, \infty)$ .

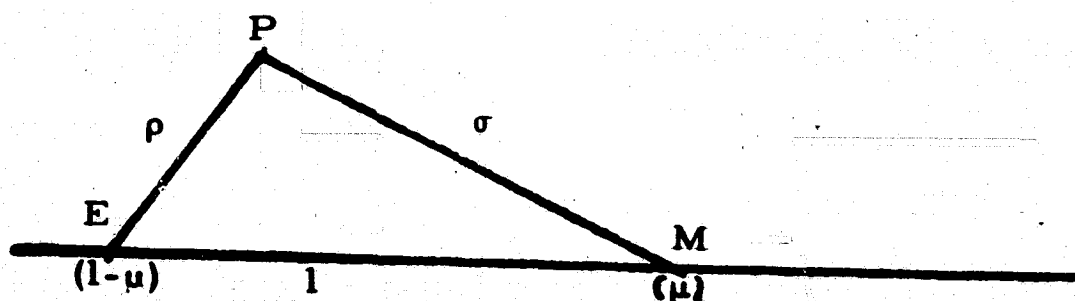


Figure 1.

Equation (1), because of its simplicity, is advantageous in computing the curves of the family. The bipolar coordinate system is also of interest because of its close relation to certain of the systems of regularized coordinates.<sup>3</sup> Thus the Thiele

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\*Superscripts refer to bibliography given at the end of paper.

transformation to coordinates  $(u, v)$  relates  $(u, v)$  to  $(\rho, \sigma)$

by the equations

$$(2) \quad \begin{aligned} \rho + \sigma &= \cosh v = \xi \\ \rho - \sigma &= \cos u = \eta \end{aligned}$$

and to Cartesian coordinates  $(x, y)$  by the equations

$$(3) \quad \begin{aligned} x &= \left( \mu - \frac{1}{2} \right) - \frac{1}{2} \cosh v \cos u, \\ y &= \frac{1}{2} \sinh v \sin u \end{aligned}$$

The zero velocity curves are well known in Cartesian coordinates; it is our purpose to determine them in bipolar and Thiele coordinate systems.

By means of the triangular inequalities in the triangle EPM it is simple to show that  $1 \leq \xi = \rho + \sigma$  and that  $-1 \leq \eta = \rho - \sigma \leq 1$ .

The region of interest is thus a rectangular semi-infinite strip in the  $(\xi, \eta)$ -plane. It becomes a corresponding rectangular strip in the  $(\rho, \sigma)$ -plane by a  $45^\circ$  - rotation and contraction by a factor  $\sqrt{2}$ . The curves are therefore qualitatively similar in the  $(\rho, \sigma)$ - or  $(\xi, \eta)$ -planes. They are shown in Figure 2 for  $\mu = 1/80$ .

The curves  $C_2$ ,  $C_4$ , and  $C_6$ , passing through three critical points  $R_2$ ,  $R_1$ , and  $R_3$ , respectively, divide the region in the manner of separatrices. The curve  $C_2$  separates members of the family which intersect the line  $\xi = 1$  (the segment of the earth-moon line between E and M), from those which do not. The curve  $C_4$

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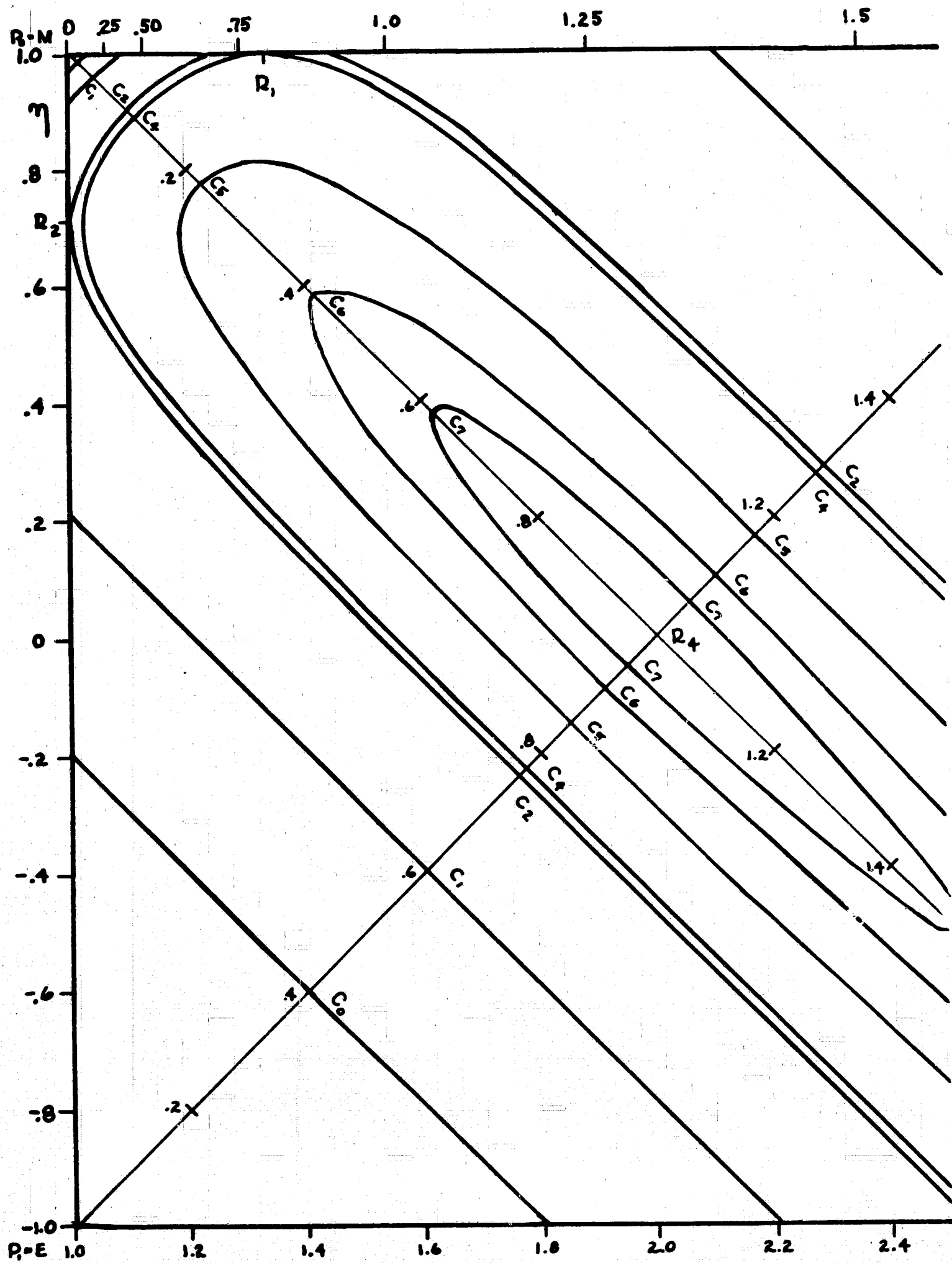


Fig. 2 (sheet 1)

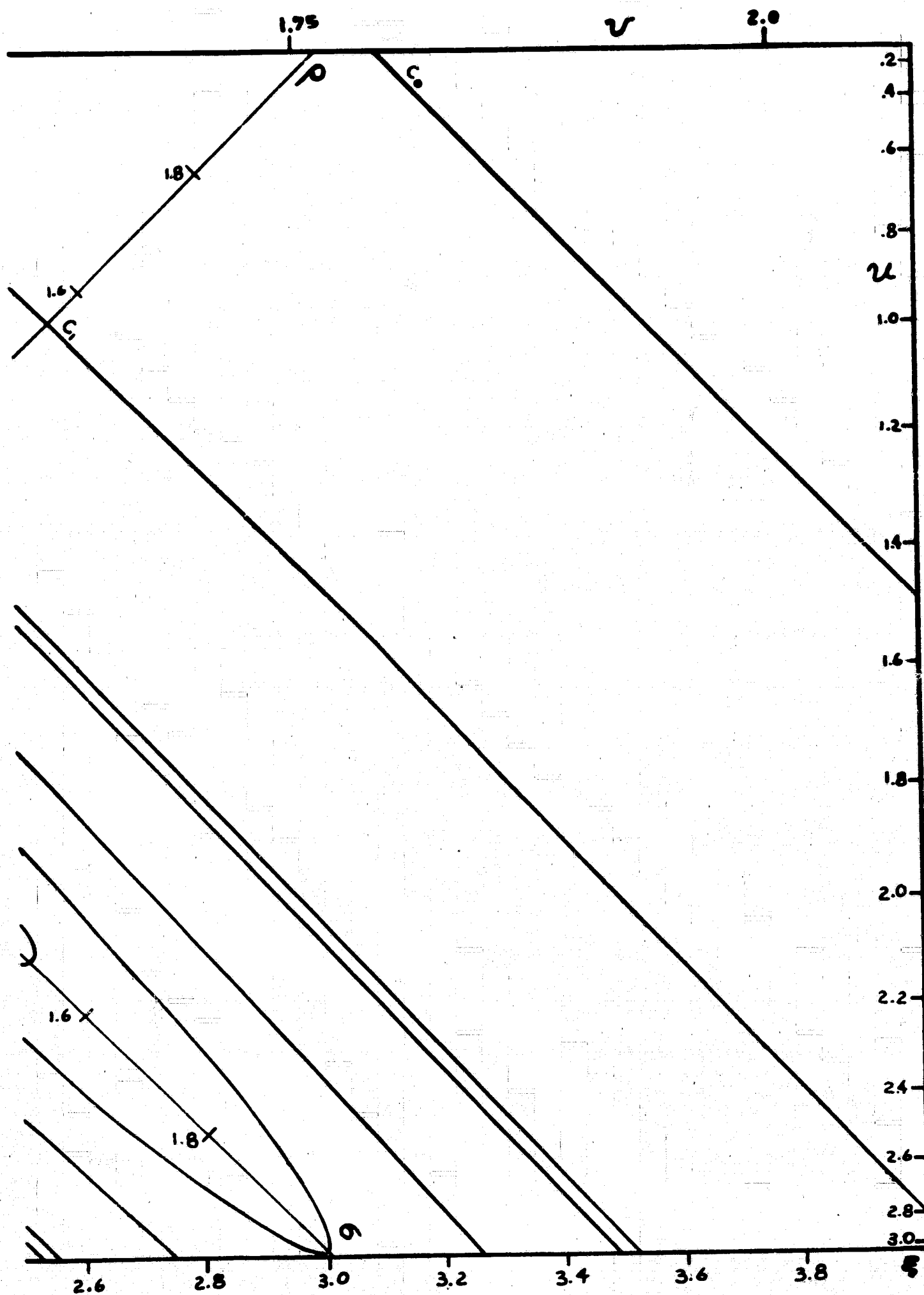


Fig. 2 (sheet 2)

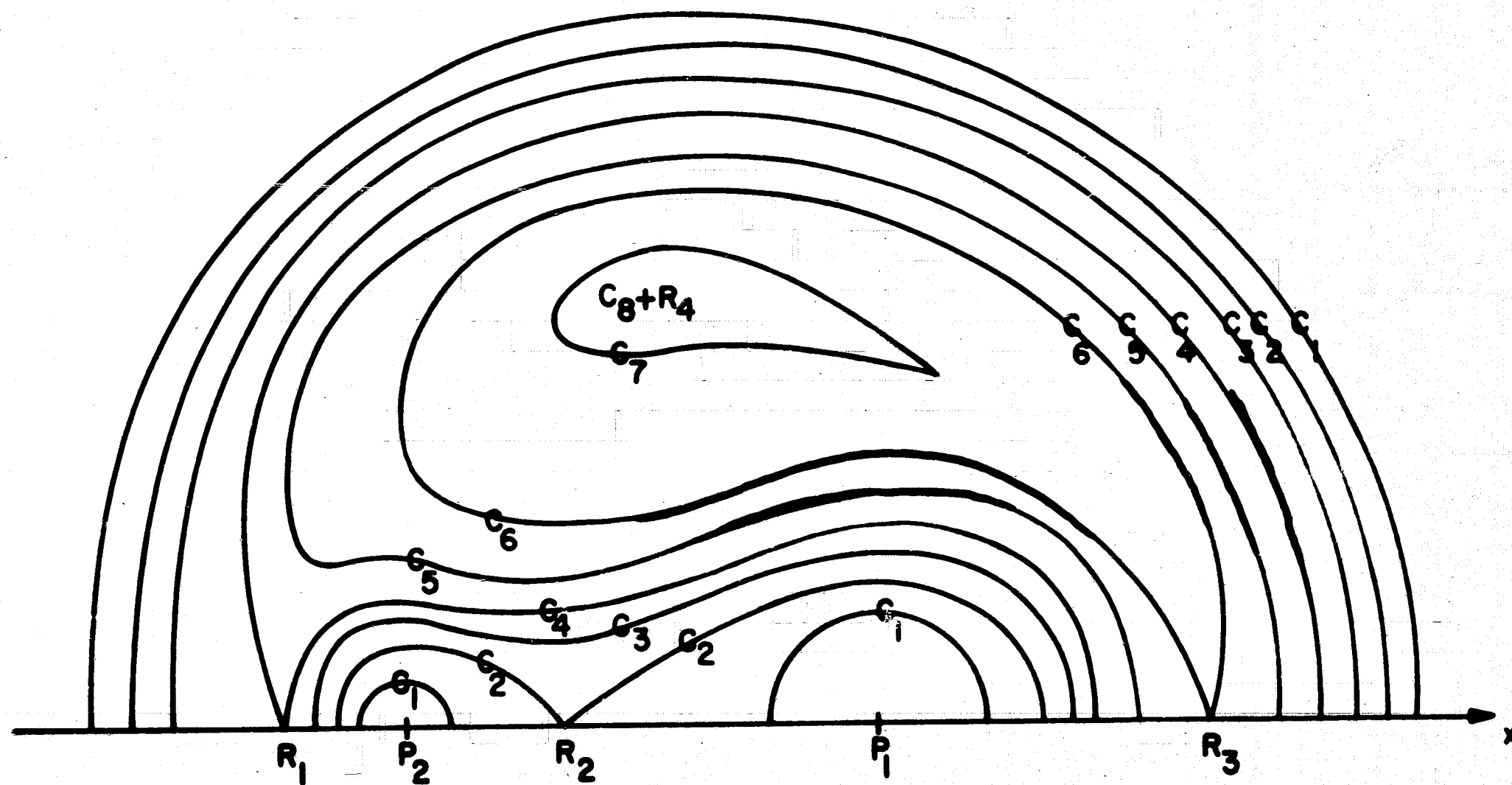


Figure 3

separates members of the family which intersect the line  $\eta = +1$  ( the segment of the earth-moon line beyond M ) from those which do not. And the curve  $C_6$  separates members of the family which intersect the line  $\eta = -1$  ( the segment of the earth-moon line beyond E ) from those which do not. Other, non-critical curves are also shown. The curve  $C_8$  is the single point  $R_4$ , the Lagrangian point  $\rho = 1 = \sigma$  and its mirror image  $R_5$ , at which  $C$  achieves its minimum value of 3. The curves are thus arranged in the order of values of  $C$  as  $3 = C_8 < C_7 < C_6 < C_5 < C_4 < C_2 < C_1 < C_0$ . ( No curve  $C_3$  is given since curves  $C_4$  and  $C_2$  lie so close together. ). These curves are to be compared directly with those given in Cartesian coordinates by Szebehely, reproduced here as Figure 3. The labelling of points and figures in Figure 2 has been made to correspond to those of Figure 3. Equations (2) permit the employment of either the  $(\xi, \eta)$  variables or the  $(u, v)$  variables according to the scale chosen along each axis. At the same time, axes in the  $(\rho, \sigma)$  system have been indicated on the same figure.

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3. Ibid., pp. 22-35, esp. pp. 34-35.

# INVESTIGATIONS OF THE EULER PROBLEM BY GRAPHICAL ANALYSIS

The representation of the Euler Problem in the bipolar coordinate system is now well known. The equations of motion in this coordinate system are

$$(1) \quad \frac{d^2(\rho^2)}{dt^2} = 4\epsilon + \frac{2\mu}{\rho} + \frac{(1-\mu)}{\sigma^3} [3\sigma^2 + a^2 - \rho^2]$$

$$\frac{d^2(\sigma^2)}{dt^2} = 4\epsilon + \frac{2(1-\mu)}{\sigma} + \frac{\mu}{\rho^3} [3\rho^2 + a^2 - \sigma^2]$$

where  $\rho$  and  $\sigma$  are the distances from the two fixed bodies  $E$  and  $M$  ( of mass  $\mu$  and  $(1-\mu)$  respectively ) to the moving body of negligible mass,  $a$  is the constant distance between  $E$  and  $M$ , and the constant  $\epsilon$  is energy given by

$$\epsilon = \frac{1}{2} m \left( \frac{ds}{dt} \right)^2 - \left[ \frac{\mu}{\rho} + \frac{1-\mu}{\sigma} \right]$$

where

$$\frac{ds}{dt} = \frac{\mu}{\sin \alpha} \sqrt{1 - 2(\cos \alpha) \frac{d\sigma}{d\rho} + \left( \frac{d\sigma}{d\rho} \right)^2},$$

$m$  is mass, and  $\alpha$  is the included angle between  $\rho$  and  $\sigma$ . For the problem at hand  $\epsilon$  is bounded by  $-1.2 < \epsilon < 0$ . The above

system, (1), rewritten in a  $(\xi, \eta)$  coordinate system, where

$$\xi = \frac{1}{2} (\rho + \sigma), \text{ and } \eta = \frac{1}{2} (\rho - \sigma), \text{ and with } a = 2, \text{ yields}$$

$$(2) \quad \frac{8[dt]^2}{[\xi^2 - \eta^2]^2} = \frac{[d\xi]^2}{[\xi^2 - 1][\epsilon \xi^2 + \xi - \gamma]} = \frac{[d\eta]^2}{[\eta^2 - 1][-\epsilon \eta^2 - 2(2\mu - 1)\eta + \gamma]}$$

where  $\gamma$  is a constant of integration. With the condition that

$-1.2 < \epsilon < 0$  the bounds on  $\gamma$  are given by

$$0 < \epsilon + 2(2\mu - 1) \leq \gamma \leq \epsilon \xi_1^2 + 2\xi_1$$

where  $\xi_1$  is the largest value which  $\xi$  is permitted to assume.

The region of interest is given in Figure 1.

The above system of differential equations has been studied in certain instances by the use of the Jacobi Elliptic Functions to reveal some properties of the trajectories which this system represents. The primary objective of current investigations is to examine this system through the use of the Weierstrass P-Function in order to obtain properties of the trajectories which have not been determined by the use of the Jacobi Elliptic Functions. A graphical analysis should render possible general trends of the trajectories which will aid in the future analytic scrutinization.

The first two members of (2) are transformed into the  $(u, v)$  coordinate system by using

$$\xi = 1 + \frac{6(\epsilon - \gamma + 2)}{12v + (\gamma - 5\epsilon - 6)}, \quad \eta = 1 + \frac{6(\epsilon - \gamma + 4\mu - 2)}{12u + [\gamma - 5\epsilon - 6(2\mu - 1)]}$$

This transformation on (2) yields

$$\frac{dv}{\left\{ \left[ v - \frac{\epsilon + \gamma}{6} \right] \left[ 4v^2 + \frac{2(\epsilon + \gamma)}{3} v + \frac{(\epsilon + \gamma)^2}{36} - \delta_1 \right] \right\}^{1/2}} = \frac{du}{\left\{ \left[ u - \frac{\epsilon + \gamma}{6} \right] \left[ 4u^2 + \frac{2(\epsilon + \gamma)}{3} u + \frac{(\epsilon + \gamma)^2}{36} - \delta_2 \right] \right\}^{1/2}}$$

where  $\delta_1 = \epsilon\gamma + 1$  and  $\delta_2 = \epsilon\gamma + (2\mu - 1)^2$ .

Each of the cubics which appear in the radicand of each member of this differential equation is of the form  $p(z) = 4z^3 + g_2z + g_3$ .

For  $z = u$ , then

$$g_2 = - \left[ \frac{(\epsilon + \gamma)^2}{18} + \delta_2 \right], \quad g_3 = - \frac{\epsilon + \gamma}{6} \left[ \frac{(\epsilon + \gamma)^2}{36} - \delta_2 \right]$$

and, for  $z = v$ , we have

$$g_2 = - \left[ \frac{(\epsilon + \gamma)^2}{18} + \delta_1 \right], \quad g_3 = - \frac{\epsilon + \gamma}{6} \left[ \frac{(\epsilon + \gamma)^2}{36} - \delta_1 \right]$$

The roots of these cubics can readily be obtained.

For the case  $z = u$ , we get

$$u_1 = \frac{\epsilon + \gamma}{6}, \quad u_2 = \frac{1}{2} [-u_1 - \sqrt{\delta_2}], \quad u_3 = \frac{1}{2} [-u_1 + \sqrt{\delta_2}]$$

and in the case of  $z = v$ , we have

$$v_1 = \frac{\epsilon + \gamma}{6}, \quad v_2 = \frac{1}{2} [-v_1 - \sqrt{\delta_1}], \quad v_3 = \frac{1}{2} [-v_1 + \sqrt{\delta_1}]$$

Here, and throughout, the restriction  $\delta_2 > 0$  is made.

(Note that since  $0 < \mu < 1$ , then  $\delta_1 > \delta_2$ .) With this assumption the cubics have three real roots. Since  $g_2 < 0$ , then, from Descartes' Rule of Signs, the cubic has two positive and one

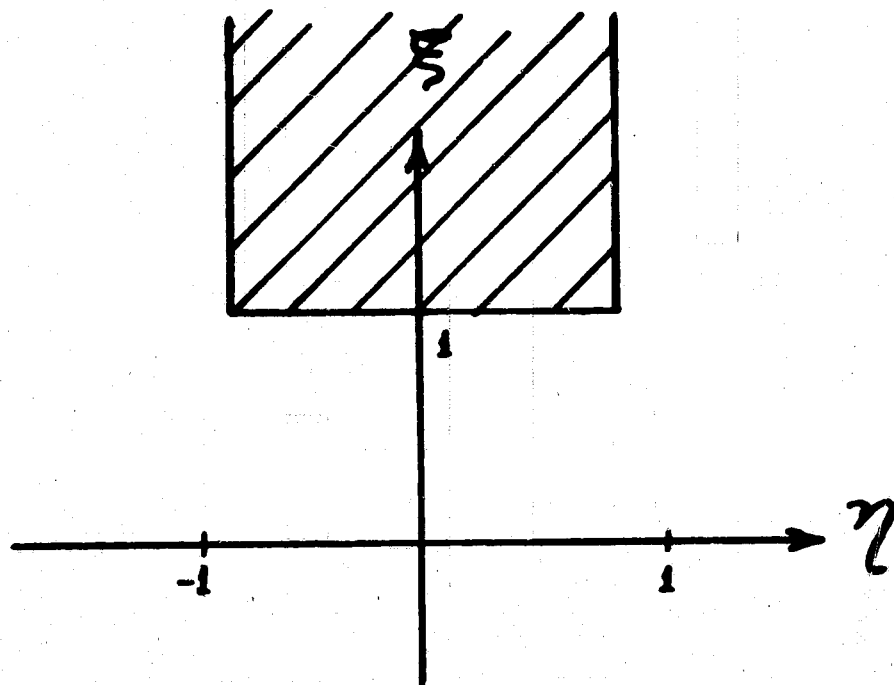


Fig. 1

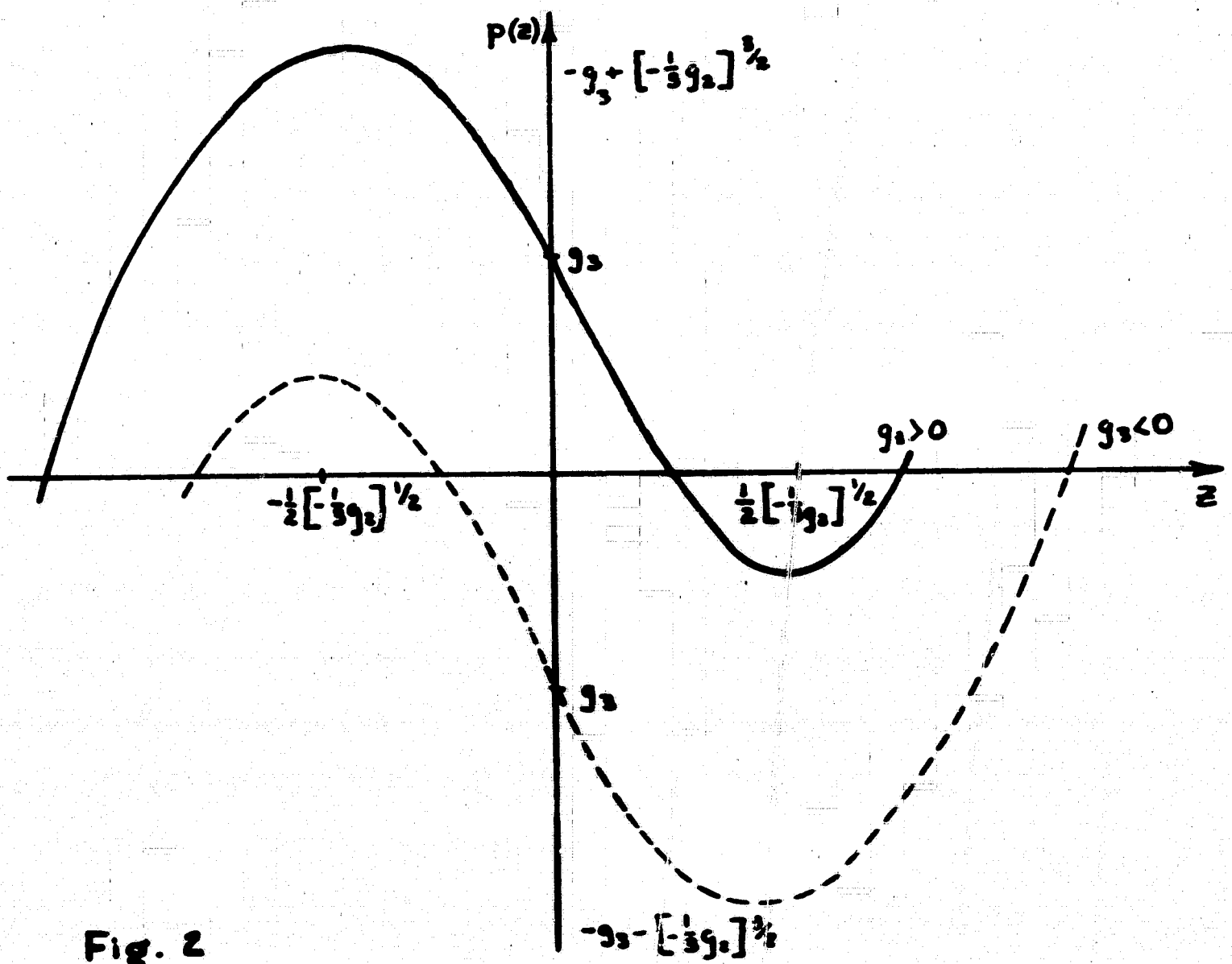


Fig. 2

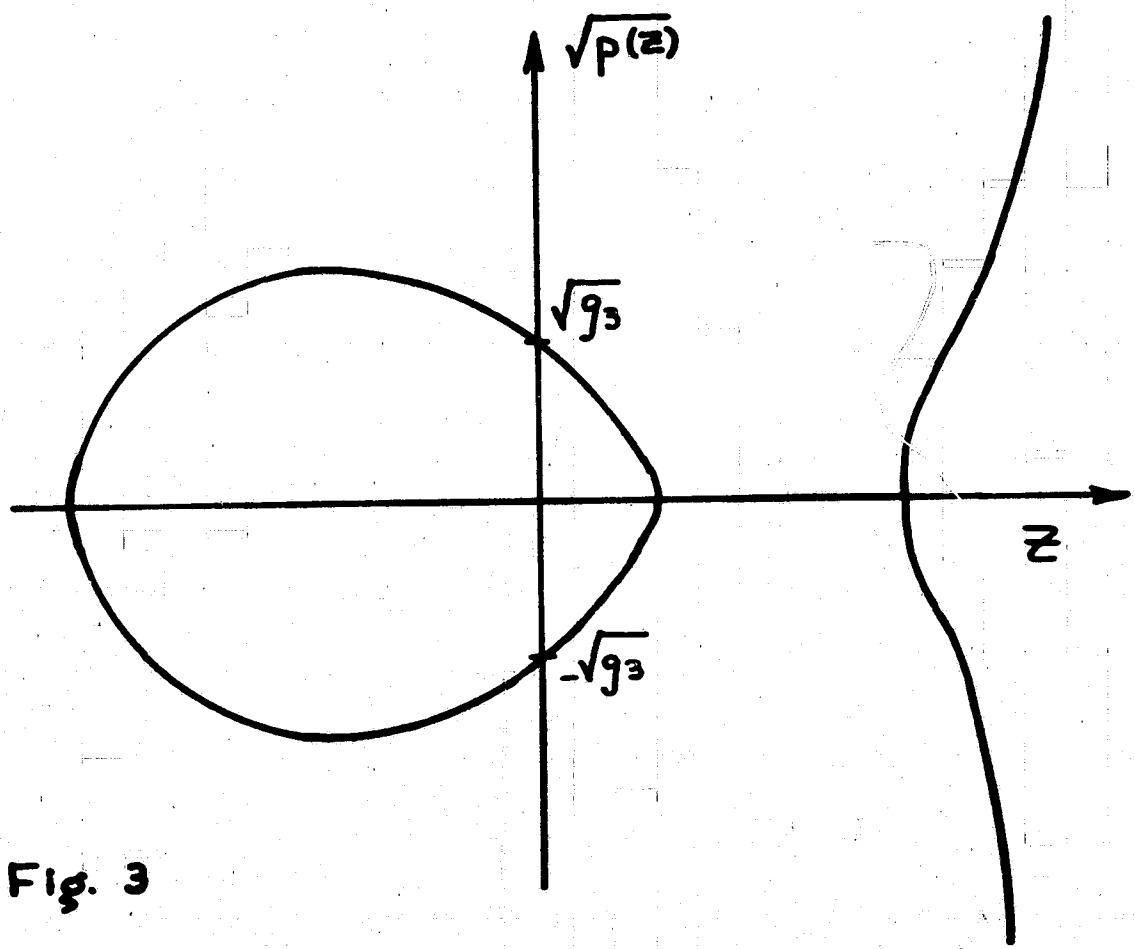


Fig. 3

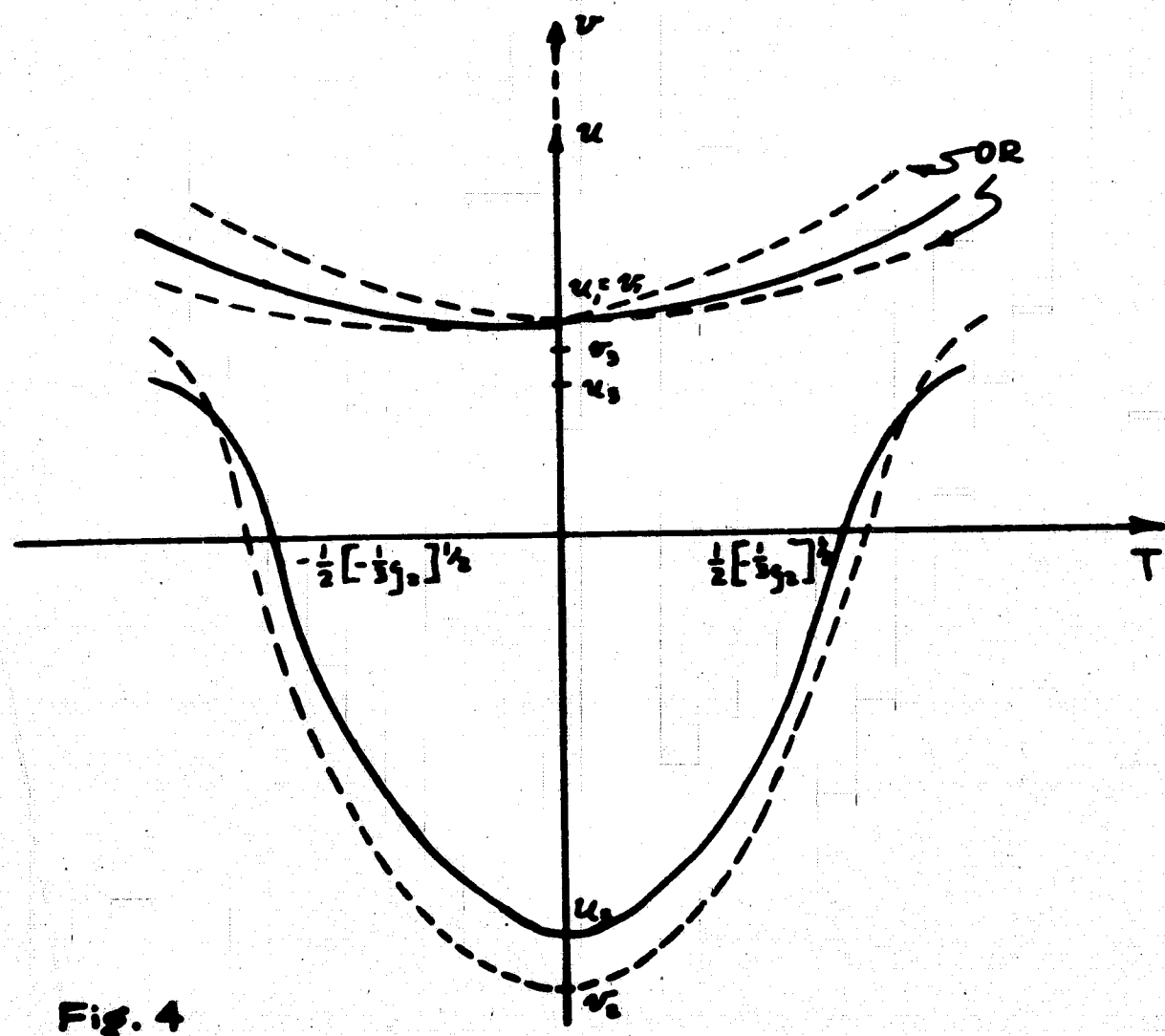


Fig. 4

negative root if  $g_3 > 0$ . The reverse situation prevails for  $g_3 < 0$ . The general shape of these curves is given in Figure 2. From the values of the roots it follows that  $u_2 < u_3$  and that  $v_2 < v_3$ . Additional assumptions and restrictions help to isolate other properties of the roots.

Since the denominators of the differential equations of interest contain  $\sqrt{p(z)}$ , then it is desirable to study  $\sqrt{p(z)}$  against  $z$ . The general form of  $\sqrt{p(z)}$  sketched against  $z$ , and for  $g_3 > 0$ , is given in Figure 3.

A still further graphical analysis can be made if each member of the last given equation is equated to  $dT$ . Thus,

$$dT = \frac{2\sqrt{2} dt}{\xi^2 - \eta^2}$$

yields

$$\frac{du}{dT} = \sqrt{p(u)}, \quad \frac{dv}{dT} = \sqrt{p(v)}.$$

Furthermore,  $dT > 0$  for  $\xi > 1$  and  $-1 < \eta < 1$ . The curve depicted in Figure 3 may now be interpreted as the slope of the curve in the  $u, T$ - or  $v, T$ -phase plane. An analysis of the behavior of these slopes can be obtained from Figure 3. The results of this analysis yields curves of the general form given in Figure 4.

The purpose of this study is to examine the trajectory in the  $u, v$ - phase plane through a graphical analysis. This is accomplished in the  $u, v$ - coordinate systems in either of two ways. First, we

consider curves which represent the slopes  $\frac{du}{dT}$  and  $\frac{dv}{dT}$  as indicated in Figure 3 to obtain  $\frac{du}{dv}$ . This is to be followed by an analysis of the curve of these slopes to obtain the general form of the trajectory in the  $u, v$ - plane. The second method is to eliminate the parameter  $T$  from the function  $u$ , and  $v$  as displayed in Figure 4, to obtain the general form of the trajectory in the  $u, v$ -plane. Both of these methods prove to be too unruly to obtain specific and definite results. This seems to be due to the choice of the coordinate system which is used and not particularly to the methods pursued. A new coordinate system is being sought with which one may extract simpler and more exact analyses.

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**NUMERICAL APPROXIMATION OF MULTIVARIATE FUNCTIONS  
APPLIED TO THE ADAPTIVE GUIDANCE MODE**

By

**R. J. Vance**

**Detroit, Mich.**

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A system of orthonormal polynomials for approximating two control functions is described. Properties that are useful in formulating computational methods are also given. Theorems of functional analysis that are relevant to the selection of data points are stated.

INTRODUCTION

The implementation of the Adaptive Guidance Mode requires the development of two control functions for a specific mission by a space vehicle. Numerical methods offer a direct way of obtaining polynomial or rational approximants of these functions from tabulated data of minimum fuel trajectories. Many criteria are available to measure the "goodness" of the approximation. Corresponding to each error criterion are one or more methods of approximation. For example, normal equations or orthogonal polynomials may be used to obtain at least squares approximation, while linear programming or special algorithms may be used to achieve a Tchebychev approximation.

However, in the special application to the Adaptive Guidance Mode, other criteria are superimposed. The most important of these is the consumption of fuel in excess of the minimum in order to complete the mission in the presence of various perturbations during the flight of the vehicle. There is no simply expressed relation between this fuel error and the error of approximation, but decreasing the error of the approximation will decrease the excess of fuel used. If the approximants to the control functions have high accuracy, then the minimum fuel characteristic of the flight path will follow.

There are two basic problems in the approximation of a function of many variables:

- (i) The selection of the "best" model as an approximant to the function in question.
- (ii) The selection of the "best" set of tabulated data for use in the numerical approximating procedure.

The first problem is handled well in the present application when a linear combination of orthonormal polynomials is used as a least squares approximant. The uniqueness and existence of solutions, and a multitude of other questions, have been answered very well during the past seventy years. Questions concerning the selection of data points seem to be more resistant to direct methods and call for more powerful tools such as functional analysis, combinatorial topology, and the theory of functions of many variables. During the past ten years, a few theorems have been formulated and proved which point the way for the solution of this problem.

#### A. Description of the Problem

For a specific mission (end point or final state of the vehicle) and given initial conditions, a minimum fuel trajectory may be generated by applying the calculus of variations. A convenient way to achieve this is through a numerical solution to the calculus of variations problem via a digital computer program. The optimum values of the steering and cutoff functions at regular time intervals along this path are then given. By developing a number of such trajectories, representing all possible disturbances which can affect vehicles of the Saturn class and still result in the achievement of the specific mission, the values of the path parameters, to be used in the approximation, are tabulated (see Appendix I for a list of the parameters and units used). Then, the control functions to be approximated can be thought of as 8-tuples, e.g.,  $(X, t, \frac{F}{w}, y, \dot{y}, x, \dot{x}, \frac{m}{m})$  for the steering function  $X$  and cutoff function  $T_r$ .

Since the range of values which can be assumed by the 8 parameters is finite, we may think of the functions as a mapping from a 7-dimensional interval in euclidean space (the domain  $D$  of  $(t, \frac{F}{w}, y, \dot{y}, x, \dot{x}, \frac{m}{m})$ ) to a segment of the real line (the range of  $X$  or of  $T_r$ ). Now consider the linear space  $F$  of all continuous functions of these 7 parameters and the subspace  $P$  of all polynomials of a given degree in the parameters defined on the same domain  $D$ . Now  $X$  or  $T_r$  are in  $F$  but not necessarily in  $P$ . The functions  $f \in F$  may be thought of as points in  $F$ .

The problem of approximating  $X$  or  $T_r$  is one of finding a set of points  $\{P_j\} \in P$ , which is closest to the point  $X$  or  $T_r$ , respectively, in terms of some metric or "measure of error." If the  $L^{(2)}$  norm (least squares approximation) is used as the metric, then there is a unique point  $P_j \in P$  which is closest to the point  $X$  or  $T_r$ . However, if the Tchebychev Norm ( $\max |X - P_j|$  over domain  $D$ ) is used as a metric, then there may be many points which are "closest" to  $X$  or  $T_r$ . One might say that we have an uncountable convex set of best approximations.

## B. ORTHOGONAL POLYNOMIALS AND APPROXIMATION

The use of a system of orthogonal polynomials simplifies the approximation of a function while controlling errors introduced by computations. Moreover, there is a large amount of literature concerning this subject and its relations with convergent series, linear spaces, unique approximations, projection operators, Hilbert Space, convex sets and other mathematical objects relevant to approximation theory. This makes it possible to obtain much information with a minimum of computation.

For approximating the control functions,  $X$  and  $T_r$ , we are given the following:

1. A selected set  $S$  and  $n$  data points. This set will vary with the function to be approximated and the approximants used. If  $s_i \in S$ , then  $s_i$  is of the form  $s_i = (x_i, t_i, (\frac{F}{w})_i, x_i, \dot{x}_i, y_i, \dot{y}_i, (\frac{\dot{m}}{m})_i)$  where the subscript  $i$  indicates the 8-tuple is evaluated at the  $i$ th data point.

2. A set  $B$  of basis functions  $b_j$  used in a linear (or perhaps non-linear) combination as approximants to  $X$  or  $T_r$ . If a polynomial approximation is the desired goal then the basis functions would be monomials or polynomials in the last 7 components of  $s$ . In any case, the values of the basis functions can be determined from the values of the components.

Before proceeding, the following definitions are cited:

(i) The inner product  $(b_j, b_k)$  of two functions is defined by

$$(b_j, b_k) = \sum_{i=1}^n \gamma_i b_{ji} b_{ki}$$

where  $i$  indicates the value of the function at the  $i$ th data point.  $\gamma_i$  is a weight associated with the point  $i$ . The weights assigned to data points will be discussed in Section C.

(ii) Two functions  $b_j, b_k$  ( $j \neq k$ ) are said to be orthogonal if  $(b_j, b_k) = 0$ .

(iii) A function  $b_j$  is said to be normalized if  $(b_j, b_j) = 1$ .

(iv) A system of functions  $b_j$  is said to be orthonormal if (ii) is true for every member with respect to every other member and if (iii) is true for every member. To form an orthonormal system  $\{q_j\}$  of polynomials, step by step, from a set of basis functions  $\{b_j\}$ , we begin by letting  $b_1 = \underline{1}$ , where the bar under the 1 indicates a function 1 which is constant ( $= 1$ ) over all the  $n$  data points. Normalizing  $b_1$ , we obtain  $q_1$ :

$$q_1 = \frac{b_1}{\sqrt{n}} = \frac{\underline{1}}{\sqrt{n}}$$

so

$$(q_1, q_1) = \sum_{i=1}^n \frac{1_i}{\sqrt{n}} \cdot \frac{1_i}{\sqrt{n}} = 1.$$

We will assume here that  $\gamma_i = 1$  for all  $i$ .

Now, let  $b_2 = x$  and form a polynomial  $p_2$  orthogonal to  $\frac{\underline{1}}{\sqrt{n}}$ .

$$p_2 = x - \underline{1} \sum_{i=1}^n \frac{x_i}{\sqrt{n}} \cdot \frac{1_i}{\sqrt{n}} = x - \frac{1}{n} \sum_{i=1}^n x_i$$

$$(p_2, \frac{1}{\sqrt{n}}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ x_i - \left( \frac{1_i}{n} \sum_{i=1}^n x_i \right) \right]$$

$$= \frac{1}{\sqrt{n}} \left[ \sum_{i=1}^n 1_i x_i - 1 \sum_{i=1}^n x_i \right]$$

$$= \frac{1}{\sqrt{n}} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n x_i \right) = 0$$

$q_2$  is obtained by normalizing  $p_2$

$$q_2 = \frac{p_2}{(p_2, p_2)^{\frac{1}{2}}}$$

A convenient way of generating polynomials orthogonal over the data is given by adjoining a row and column to Gram's determinant. Let  $p_{h+1}$  be the  $(h+1)$ th orthogonal polynomial generated by using the basis function  $b_{h+1}$ , then

$$p_{h+1} = \begin{vmatrix} (b_1, b_1) & (b_1, b_2) & \cdot & \cdot & \cdot & (b_1, b_{h+1}) \\ (b_2, b_1) & (b_2, b_2) & \cdot & \cdot & \cdot & (b_2, b_{h+1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (b_h, b_1) & (b_h, b_2) & \cdot & \cdot & \cdot & (b_h, b_{h+1}) \\ b_1 & b_2 & & & & b_{h+1} \end{vmatrix}$$

This determinant is easily expanded by its last row of basis functions; since all other rows consist of numbers.  $p_{h+1}$  is seen to be orthogonal to every basis function  $b_j$  ( $j = 1, 2, \dots, h$ ) since taking the inner product  $(p_{h+1}, b_j)$  is equivalent to forming a determinant whose last row is identical to its  $j$ th row.  $p_{h+1}$  is then orthogonal to any linear combination of the  $b_j$  and, therefore, to every  $p_j$  and  $q_j$  ( $j < h+1$ ). By normalizing, we have

$$q_{h+1} = \frac{p_{h+1}}{(p_{h+1}, p_{h+1})^{\frac{1}{2}}}.$$

Using the above system of orthonormal polynomials, the functions  $X$  and  $T_r$  may be expanded in the form

$$X \approx \sum_{j=1}^r c_j q_j.$$

$$T_r \approx \sum_{j=1}^r c'_j q_j.$$

The  $c_j$ 's and  $c_j'$ 's are Generalized Fourier Coefficients, relative to the system  $\{q_j\}$ , of  $\chi$  and  $T_r$  respectively:

$$c_j = (\chi, q_j) \quad c_j' = (T_r, q_j)$$

This orthonormal system and the expansion of the guidance functions by it have many useful properties. We list a few:

(i) The system  $\{q_j\}$  is complete; i.e., any continuous function  $u$  can be approximated in the mean (least squares) to any desired degree of accuracy by choosing a sufficient number of orthonormal polynomials to expand  $u$ . We actually have much stronger conditions that assure us of more than just approximation in the mean. The generalized Stone-Wierstrass Theorem states that there is uniform approximation by polynomials.

(ii) Bessel's inequality is true

$$\sum_{i=1}^n x_i^2 \geq \sum_{j=1}^r c_j^2.$$

From this we see the magnitude of the  $c_j$  is an indication of the importance of  $q_j$  and, therefore, of  $b_j$  in the expansion of  $\chi$ . This is clear if we remember that  $|q_j| \leq 1$  for all  $j$  and that  $|c_j| \rightarrow 0$  as  $j \rightarrow \infty$ . As

a result, the partial sums  $S_r = \sum_{j=1}^r c_j q_j$  becomes asymptotically dependent as  $r \rightarrow \infty$ .

If normal equations are used in obtaining an approximation, the matrix formed becomes ill-conditioned when sufficient terms are used. However, the matrix usually becomes ill-conditioned or singular from rounding during computations before this. In practice, if any of the  $c_j$ 's are of the order of magnitude of the roundoff errors, then the terms in the expansion containing them are deleted.

(iii) The approximation  $\chi \approx \sum_{j=1}^r c_j q_j$  is a least squares

approximation. An outline of the proof is as follows (see reference 1 for an extensive account):

Suppose

$$(1) \quad \hat{x} = \sum_{j=1}^r c_j q_j = v_r$$

with the  $c_j$ 's obtained as above. If any other set of coefficients  $d_j$  is used, the approximation will be

$$(2) \quad \hat{x} = \sum_{j=1}^r d_j q_j = w_r$$

where  $d_j \neq c_j$  for at least one  $j = 1, 2, \dots, r$ . Then we may take the squares of the differences

$$\begin{aligned} \sum_{i=1}^n (x_i - w_{ri})^2 &= (x - w_r, x - w_r) \\ &= (x - v_r - (w_r - v_r), x - v_r - (w_r - v_r)) \\ (3) \quad &= (x - v_r, x - v_r) + 2(x - v_r, v_r - w_r) \\ &\quad + (v_r - w_r, v_r - w_r). \end{aligned}$$

Bessel's inequality can now be used

$$(x, x) = \sum_{i=1}^n x_i^2 \geq (v_r, v_r) = \sum_{j=1}^r c_j^2$$

or under certain conditions Parseval's formula

$$(x, x) = \sum_{j=1}^{\infty} c_j^2.$$

We can see from the definition of inner product and Bessel's inequality (3) above is minimized only when  $w_r = v_r$ .

(iv) The expansion coefficients  $c_j$  are unique by the Reisz-Fischer Theorem which states (ref. 1, pp. 13-18):

Let  $\{q_j\}$  denote an arbitrary orthonormal system and  $\{c_j\}$  a sequence of real numbers. A necessary and sufficient condition that  $\{c_j\}$  be the sequence of expansion coefficients of an  $L^{(2)} \gamma$ -integral (Lebesgue square integral with measure  $\gamma$ ) function  $X$ , is

$$\sum_{j=0}^{\infty} c_j^2 < \infty.$$

The partial sums  $S_n$  of the expansion of  $X$  then converge to  $X$  in the sense of the  $L^{(2)} \gamma$ -metric, i.e.,  $\lim_{n \rightarrow \infty} \left\{ \int \dots \int_D (X - S_n)^2 d\gamma \right\}^{\frac{1}{2}} \rightarrow 0$ .

(v) Many authors speak of the coefficients  $c_j$  as the coordinates of the function  $X$  in a Hilbert space with a countable orthonormal basis  $\{q_j\}$ . As an immediate consequence, we have that any set of the orthonormal polynomials with their expansion coefficients of a given function  $X$  is an approximation.

If a polynomial approximation of  $X$  is desired, having only specified basis functions, then one adds only those terms in the expansion of  $X$  which contain the specified basis functions.

(vi) The approximation of a function  $X$  by a ratio of polynomials may be accomplished by the use of an orthonormal system. Suppose, we wish an approximation of the form

$$X \approx \frac{x + d y}{a x^2 + b y^2 + h}. \quad (1)$$

Transforming this into a linear relation, we have

$$\frac{x}{X} = a x^2 + b y^2 + h + d \left( \frac{-y}{X} \right). \quad (2)$$

Designating  $\frac{x}{X}$  as the function to be approximated and  $x^2$ ,  $y^2$ , 1, and  $\frac{-y}{X}$  as the basis functions, a first approximation results by division and multiplication of terms in (2)

$$\chi \approx \frac{x + d_1 y}{a_1 x^2 + b_1 y^2 + h_1}.$$

A better approximation is insured if one more step is taken. Let  $\frac{x + d_1 y}{\chi}$  be the new function to be approximated by the basis functions  $x^2$ ,  $y^2$ , and 1. This will be in the form

$$\frac{x + d_1 y}{\chi} \approx a_2 x^2 + b_2 y^2 + h_2.$$

Finally, we have

$$\chi \approx \frac{x + d_1 y}{a_2 x^2 + b_2 y^2 + h_2}.$$

The process may be repeated several times for the determination of coefficients in both the numerator and denominator.

### C. THE SELECTION OF DATA POINTS

Consider the functions we are approximating as ordered  $(r + 2)$ -tuples

$$u_i = (\chi_i, 1, b_{1i}, b_{2i}, \dots, b_{ri})$$

where  $i = 1, 2, \dots, n'$  and  $n'$  is the total number of data points generated.  $b_{ji}$  is the  $j$ th basis function evaluated at the  $i$ th point. Define the position function  $g_j$  by

$$g_j u_i = g_j (\chi_i, 1, b_{1i}, b_{2i}, \dots, b_{ji}, \dots, b_{ri}) = b_{ji}.$$

Now form the set  $\{u_i: g_j u_i = \text{MAX or MIN}; j=1, 2, \dots, r; i = 1, 2, \dots, n'\} = \{u_i^*\}$ .  $\{u_i^*\}$  may be thought of as the vertices of a polytope in a space of  $(r + 2)$  dimensions. Any point  $u_i$  of the set  $\{u_i\}$  not in  $\{u_i^*\}$ , i.e.,  $u_i \in \{u_i\} \setminus \overline{\{u_i^*\}}$ , can be written as

$$u_i = \sum_{k=1}^L a_k u_k^*$$

where

$$\sum_{k=1}^L a_k = 1 \quad a_k \geq 0 \quad (k = 1, 2, \dots, L).$$

There are  $L$  points in  $\{u_i^*\}$ .

The  $a_k$ 's are called the barycentric coordinates of the point  $u_i$  and the  $u_i$ 's are interior points of a convex set. Using these coordinates, the error  $e_i$ , associated with  $u_i$ , may be written in terms of the errors  $\{e_k^*\}$  at the points  $\{u_k^*\}$ :

$$e_i = \sum_{k=1}^L a_k e_k^*.$$

Knowing this, a program can be written to obtain the set  $\{u_i^*\}$  and, through linear programming, a bound on the errors of an approximation using specific basis functions can be computed. This same result also determines the coefficients in a polynomial approximation. That is, a linear programming problem is set up of the form:

$$\text{Minimize } W_i - Z_i = \frac{1}{e_i^*} \left( X_i - \sum_{j=0}^r \alpha_j b_{ji} \right) \quad i = 1, 2, \dots, L$$

$$\text{subject to } W_{n+j+1} = \alpha_j$$

$$\frac{1}{e_i^*} X_i = W_i - Z_i + \sum_{j=0}^r \frac{1}{e_i^*} (W_{n+j-1} - Z_{n+j+1}) b_{ji}$$

where  $e_i^*$  is the error at  $u_i^*$ ,  $b_{ji}$  the basis function  $b_j$  evaluated at this same point, and  $W_i, Z_i$  are non-negative parameters used in linear programming to determine the coefficients  $\alpha_j$ . The  $e_i^*$  may be preassigned and the resulting system tested for consistency.

In defining the inner product of two function  $b_j, b_k$  in  $B(i)$  by a weighted sum, we are representing an integral over a domain on which the seven state parameters  $(t, x, \dot{x}, y, \dot{y}, \frac{F}{w}, \frac{m}{m})$  are defined.

$$\sum_{i=1}^n \gamma_i b_{ji} b_{ki} \approx \int \dots \int_D b_j b_k d_\mu \left( x, y, \dot{x}, \dot{y}, t, \frac{F}{w}, \frac{m}{m} \right).$$

The  $\gamma_i$  are volumes of 7-dimensional parallelepipeds. The  $b_j$  are evaluated at some interior point of them in a manner analogous to the use of the mean value theorem and rectangles to evaluate integrals in two dimensions. A method of selecting the sample points and calculating the proper  $\gamma_i$  would require careful thought and work based on this geometrical interpretation. The method may not be relatively straight forward and a computer program may be written only by assuming that the values of the integrand change little in the interiors of these 7-dimensional volumes. This method is being studied at present.

We may be spared considerable work if two theorems of functional analysis are applied.

Theorem 1: (ref.4)

If  $\pi^*$  is an admissible polynomial of best approximation to  $f$  on domain  $D$  (a compact subset of euclidean  $n$ -space), then it is also the admissible polynomial of best approximation to  $f$  on a finite point set of some  $r$  points of  $D$ .  $r$  is less than or equal to the number of basis functions used.

This set of  $r$  points is characterized in the same reference. Questions concerning the application of this theorem for computations is being investigated.

By the phrase "best approximation" in Theorem 1 is meant the best Tchebychev approximation.

A relation between the Tchebychev criterion of approximation and the least squares is given by:

Theorem 2: (ref. 2, R. C. Buck p. 21).

Let  $\pi_k^*$  be the best  $L^{(k)}$  approximation to  $f$  on  $D$ , by functions  $\pi$  in a finite dimensional subspace  $M$ . Then, the uniform limits of sequences of  $\{\pi_k^*\}$ , as  $k$  increases, are all in the set of best Tchebychev approximations to  $f$ .

At first, as Buck states, Theorem 2 does not look hopeful for a computational scheme. However, in most cases, the  $L^{(2)}$  approximation is sufficiently accurate. Further relations between the Tchebychev and least squares approximations are being studied.

In summary then, a computational procedure would have the following steps:

1. Selection of basis functions to be used in the approximation.

2. Selection of the data points according to criteria based on the generalized mean value theorem or the results of Rivlin and Shapiro (ref. 4). The points selected for  $X$  will not, in general, be the same as those for  $T_r$ .

3. Formation of a system from the basis functions that is orthonormal over the data points as described in Section B.

4. Using this system, computation of the expansion coefficients of the function  $X$  or  $T_r$  which are being approximated. Bessel's inequality and the accuracy of the data used can be the basis of a criterion for discarding certain terms of the expansion as being unimportant or erroneous.

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## APPENDIX I

Parameters used in the approximation.

Parameter	Unit	Description
$t$	seconds	Real time from lift off.
$\frac{F}{W}$	lb/lb	Thrust over weight.
$\chi$	degrees	Steering parameter.
$y$	meters	Position and velocity in earth centered inertial coordinate system.
$\dot{y}$	meters/sec	
$x$	meters	
$\dot{x}$	meters/sec	
$\frac{\dot{m}}{m}$	$\frac{kg}{m} \cdot s / \frac{kg \cdot s^2}{m}$	Mass flow rate over mass.
$T_r$	seconds	Time remaining to cutoff.

## APPENDIX II

A BIBLIOGRAPHY FOR THE APPROXIMATION OF FUNCTIONS OF  
MORE THAN ONE VARIABLE

## INTRODUCTION

Both theoretical and practical aspects of approximating functions of many variables are discussed in the listed references. Only polynomials and ratios of polynomials are considered here as the approximators. Both the Tchebycheff and least squares criteria are discussed in the references extensively, while other "error" criteria are occasionally spoken of. An effort has been made to list references giving examples of approximations made by various methods and also papers which give a physical interpretation to these methods.

## SECTION I. APPROXIMATION THEORY: FUNCTIONS OF SEVERAL VARIABLES.

## A. THEORETICAL FOUNDATIONS

## REFERENCES

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Sard (7, 8, 9, 11 - p 700) has done much in the area of error estimation and expressions for the "residue" in terms of functionals. His work on projection operators and variance (8, 9) are immediately applicable to least squares approximations.

Berman (1) proves the impossibility of constructing a simple polynomial operator that will place a bound on the order of the terms in a "best" approximation in the Tchebycheff sense. While he considers the case of one variable functions, his results hold a fortiori in the case of functions of many variables. In the multivariate case, there are usually many "best approximations" in the Tchebycheff sense. Rice offers some theorems relevant to approximation by ratios of polynomials.

Erde'lyi (3) provides a good introduction to systems of orthogonal and bi-orthogonal systems of polynomials for approximation or expansion of functions of more than one variable.

Since tabulated data can be expressed in terms of column and row vectors, the theory of Hilbert Spaces and Spectral Theory can be used with great advantage. One immediate consideration is that the coefficients of orthogonal polynomials used as approximators are the generalized Fourier coefficients discussed in Hilbert Space Theory. Berberian (2) gives an excellent introduction to Hilbert Space Theory while Nakano is much more refined and rigorous in his extension of the theory. The book edited by Langer (4) contains many examples of the use of this theory and the theory of functional analysis as applied to actual approximation by numerical (non-analytical) methods. The extensive bibliographies in this book are also useful.

The papers edited by H. C. Thatcher, Jr., (11) are concerned with both the numerical and analytic methods of approximation of functions of many variables. Thatcher's paper in this group of articles describes a very general type of interpolation which could prove useful in the future. However, interpolation is only a very special case of approximation and the development of an approximating method based on his interpolation scheme may have difficulties. Overcoming any difficulties would be well worthwhile since one would be able to have osculatory and hyperosculatory

approximations; i.e., not only would a function be approximated but also its partial derivatives. This would help make the undulations of the approximation around the true value of the function much less in amplitude. These undulations are one of the characteristics of polynomial and rational approximation (see Part III). Thatcher's other paper (10) considers the problems of linear independence which are important in any numerical method of approximation.

## B. POINT SELECTION

### REFERENCES

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The selection of data points for approximation purposes is closely related to the construction of quadrature formulae. However, very little is available concerning problems involving more than three variables in either of the two cases. This is not due to neglect as much as it is due to lack of theoretical foundations. Since problems of "regression" involve considerations similar to those of approximation, we find discussions of point selection from the field of statistics in the articles of Arens (11) and Elfing (12).

The ideas of combinatorial topology may be applied immediately since discrete data is used in the approximation. Pontryagin (13) presents the foundations of this subject concisely and in a readable manner.

The most important papers on the selection of points and approximation theory are those of Rivlin and Shapiro (14) and Zuhovickii (15). The article of Rivlin and Shapiro has direct applications to at least three problems:

- (i) finding the "best" polynomial approximation to a given function,
- (ii) maximizing a linear functional among polynomials of a given degree,
- (iii) finding the function of least norm having a finite number of prescribed moments.

Any results of investigating problem (i) have obvious application to the present work in approximation. Solutions to problem (ii) may be used to determine error bounds.

Many approximation problems are easily put into a problem form concerning moments, and results of studying problem (iii) are then applied. Zuhovickii does this very thing. The selection of points is of concern in both papers. Rivlin and Shapiro, as does Zuhovickii, directly approach the problem through measure theory. The interpretation and application of their findings remains as a less formidable problem.

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## SECTION II. LEAST SQUARES, MATRIX OF NORMAL EQUATIONS

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For an extensive account of the least squares method with examples see Bjerhammar (1). Lotkin (5), Osborne (7), and Morduchow (6) discuss various characteristics of results obtained by using the least squares

criteria. Greville (4) combines the ideas of Bjerhammar (1) and Thatcher (I-11), which is an important result. Since the inverse of a rectangular or even a singular matrix is defined, illconditioned matrices are no longer any concern. Greville goes on to show how any information about partial derivatives may be included in the least squares approximation. The resulting approximation should be "better" than one which does not use such information.

Other approaches to the problems of ill-conditioned matrices of normal equations are discussed by Eisemann (2), Riley (9) and Roth (10). Of course, there need be no reference to normal equations for least squares approximations. The ideas of approximation may be developed in terms of projection operators. This is done by Sard (I-8).

Foster (3) takes account of round-off errors and defines an optimum inverse. Todd and Neman (11) obtain bounds on the errors when a matrix is inverted by a computer. Their paper contains many examples of error analysis based on matrices with known inverses.

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As is well known and mentioned before, one characteristic of polynomial approximation is its wave-like departure from the true value of the approximated function. For the case of functions of a single variable see Ahieser (1), but for a more general discussion that applies to the multivariate case see Zuhovickii (7).

Much difficulty with choice of terms used in the approximating polynomial can be avoided by using an orthogonal system of polynomials. Erdelyi (I-3) and Sirazdinov both have good discussions of the subject. The ideas of systems of orthogonal polynomials (also biorthogonal systems) are generalized by Dickinson (3), whose paper is useful when the polynomials used may be orthogonal to only a few others used. This idea can be used to place zeros in strategic places in the matrix of normal equations. The assignment of weights when approximating is the subject of Dzrbasyan's paper which is unusual since it considers the case of many variables.

Butler extends the well developed theory of approximation by Bernstein polynomials from the univariate case to the bivariate one. This could possibly extend to many dimensions and prove useful. For a specific example of the use of orthogonal polynomials, one can see Pings' and Sage's article (5) or one of the many texts on regression, mathematical statistics, or design of experiments.

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Ward (11) gives a simple, readable account on the use of linear programming for the immediate problem of approximations by polynomials. Loeb (7, 8) considers the approximation of functions by ratios. This seems, in most cases, to lead to a large reduction in the error. For an extensive account of an algorithm see Goldstein (5). Dickinson (3) offers a few ideas suitable in the field of rational approximation.

As introductions to the ideas of linear programming, the book by Gass (4) and the article by Good (6) are excellent. Gass develops the concepts in terms of matrices, vectors, and polytopes, avoiding the usual mass of equations, symbols, and indices.

The article by Pyne (9) is unusual in that a method for linear programming on an analog computer is described. With the great savings in money and time on an analog, the method could prove worth while even in large sized problems. The approximate solution of the analog may be refined by the digital. If the linear program is large sized, it may be solved piece-wise as described by Dantzig (1). For a physical interpretation of linear and quadratic programming, the book of Dennis is unsurpassed. Salzer (10) considers the unusual problem of using both the Tchebycheff and least squares criteria of error in approximations.

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### III. PROPOSED STUDIES IN THE FIELD OF MULTIVARIATE APPROXIMATIONS

By

James W. Hanson  
Richard J. Painter

#### SUMMARY

A proposed study of the numerical properties of functions of more than one independent variable is outlined herein.

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PROPOSED STUDIES IN THE FIELD OF MULTIVARIATE APPROXIMATIONS

By

James W. Hanson  
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SUMMARY

18697

A proposed study of the numerical properties of functions of more than one independent variable is outlined herein.

I. INTRODUCTION

In doing a search of the literature in the field of numerical analysis on the subject of multivariate approximations, a number of very obvious and critical gaps are immediately encountered. The following quotation by Henry M. Thacher, Jr. offers an excellent summary of the situation [1]. This quote is from the introduction of Reference 1 which is a report on a conference held in October, 1959 on the subject of the numerical properties of functions of more than one independent variable.

"It may be worthwhile to examine the program, not from the standpoint of the subjects covered, but rather from the standpoint of those omitted. I do not refer to the fact that linear algebra and partial differential equations have been so thoroughly discussed that an adequate treatment would require more space than this monograph affords, but rather to the authors who are not represented because they could not be found. Among these important people are:

(1) The expert on orthogonal polynomials in more than one independent variable who can describe how to select, among the infinite number of sets of such polynomials corresponding to a given region and weighting function, the one that has roots at the base points for Gauss-type quadrature formulas. Perhaps he could also tell us something that would be helpful in least squares approximation, and about minimax polynomials analogous to those by Chebyshev.

(2) The mathematician who can give us workable criteria for the existence, and hopefully for the localization, of roots of systems of non-linear algebraic and transcendental equations in several variables.

(3) The expert in approximations who can tell us how to find polynomial approximations to multivariate functions associated with either a least squares or Chebyshev norm for the error.

(4) The brother of this expert on polynomial approximations who knows all about rational approximations in several independent variables. In view of the general lack of understanding of rational approximations in a single variable, it may be some time before this man gets his Ph.D.! He certainly will be entitled to one if he solves the problem."

## II. PROBLEM STATEMENT AND PROPOSED STUDY

In attempting to obtain useful multivariate approximations for the guidance equations in the flat-earth calculus of variations problem, it seems that the most immediate question is the selection of a "best" set of "coordinate functions" to use in the approximation.

Trial and error methods do not hold much promise toward answering this question. Furthermore, since this question is intimately related to the missing subjects related in the introduction, no useful solutions are going to be found in existing works. Hence, in order to at least make an educated guess as to what these "coordinate functions" should be, the following two lines of study are proposed.

1. Make an empirical study of the data for optimal trajectories and an analytical study of the equations of motion for the flat-earth model in order to determine the properties of the associated multi-dimensional surface. Two initial points of study would be:

a. A numerical study of the partial derivatives of the dependent variables with respect to the independent variables along various trajectories. Possibly one would discover that one or more of the independent variables could be eliminated from the approximation while still remaining within the limits of error.

b. A numerical study of the variation in each of the dependent variables while all but one of the independent variables is held constant would also exhibit properties of the surfaces in question.

2. Make a study of the analytical properties of various types of "coordinate functions" in the multivariate case in order that a reasonable choice could then be made for a "coordinate function" in light of any information gained from the above study. Types of questions to be considered would be:

a. If rational (fractional) approximations are the "best coordinate functions," should the approximation be rational in  $x$ ,  $y$ ,  $z$ , ..., or only in some subset of the independent variables while polynomial in the others?

b. If a polynomial approximation is to be attempted, what powers in  $x, y, z, \dots$  should be used? Should one try

$$\begin{array}{l} 1 \quad x \quad x^2 \\ y \quad xy \quad x^2 y \\ y^2 \quad xy^2 \quad x^2 y^2 \\ \dots \end{array}$$

or

$$\begin{array}{l} 1 \quad x \quad x^2 \quad x^3 \\ y \quad xy \quad x^2 y \\ y^2 \quad xy^2 \\ y^3 \\ \dots \end{array}$$

etc.?

### III. REFERENCES

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LINEAR PROGRAMMING APPLIED TO GUIDANCE FUNCTION FITTING

By

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Shigemichi Suzuki

SUMMARY

18698

The application of linear programming to the approximation of the guidance functions in the Adaptive Guidance Mode is considered.

I. INTRODUCTION

This report describes a continuation of the investigations in [1] of the use of linear programming techniques to approximate the guidance functions in the Adaptive Guidance Mode. The notation of [1] is used in this report. Attempts to increase the speed of convergence of the Revised Simplex Method to the solution of the dual problem for the  $L_\infty$  fit are discussed.

II. THE DUAL PROBLEM

As stated in [1], a function  $f(\bar{z})$ , whose value is known at  $n$  points,  $\bar{z}_1, \dots, \bar{z}_n$ , in a multi-dimensional space, is to be approximated by a polynomial  $P(\bar{z})$  in such a way that the maximum absolute deviation,

$\max_{k=1, \dots, n} |P(\bar{z}_k) - f(\bar{z}_k)|$ , is minimized. Letting the polynomial  $P(\bar{z}_k)$  be

$$a_0 - b_0 + a_1 w_{k1} - b_1 w_{k1} + \dots + a_{m-1} w_{k,m-1} - b_{m-1} w_{k,m-1},$$

the linear programming problem is to minimize  $\epsilon$  subject to the constraints  $a_j \geq 0$ ,  $b_j \geq 0$  for  $j = 0, \dots, m-1$ ,  $\epsilon \geq 0$  and

$$\left. \begin{aligned} a_0 - b_0 + a_1 w_{k1} - \dots - b_{m-1} w_{k,m-1} - f(\bar{z}_k) &\leq \epsilon \\ a_0 - b_0 + a_1 w_{k1} - \dots - b_{m-1} w_{k,m-1} - f(\bar{z}_k) &\geq -\epsilon \end{aligned} \right\} \text{ for } k = 1, \dots, n.$$

The dual problem is to maximize the objective function,

$$\sum_{k=1}^n (-f(\bar{z}_k)) u_k + \sum_{k=1}^n f(\bar{z}_k) u_{n+k}, \text{ subject to the constraints } A'U \leq B'$$

and  $U \geq 0$ , where  $B'$  is the  $(2n+1)$ -component column vector  $(0, 0, \dots, 0, 1)$  and  $A'$  is the matrix

-1	....	-1	1	....	1
1	....	1	-1	....	-1
$-w_{11}$	....	$-w_{n1}$	$w_{11}$	....	$w_{n1}$
$w_{11}$	....	$w_{n1}$	$-w_{11}$	....	$-w_{n1}$
$-w_{12}$	....	$-w_{n2}$	$w_{12}$	....	$w_{n2}$
$w_{12}$	....	$w_{n2}$	$-w_{12}$	....	$-w_{n2}$
.		.	.		.
.		.	.		.
.		.	.		.
$w_{1,m-1}$	....	$w_{n,m-1}$	$-w_{1,m-1}$	....	$-w_{n,m-1}$
1	....	1	1	....	1

The matrix for the original problem has  $2n$  rows, while that for the dual problem has  $2n+1$  rows. Since the computation time for the linear programming routine is approximately proportional to the cube of the number of rows of the matrix, a great decrease in computation time is to be expected in solving the dual rather than the primal problem when  $n$  is much greater than  $m$ . In problems of interest in this report,  $m$ , the

number of coefficients in the polynomial  $P(\bar{z})$ , is of order 50, while  $n$ , the number of points at which  $P(\bar{z})$  is fitted to the given function  $f(\bar{z})$ , is of order 1500. The results of a test computation did not confirm the anticipated decrease in computing time for the dual problem. A possible explanation of this result follows.

The large number of zero components in the vector  $B'$  appears to cause the slow convergence of the method. Let the following array denote the Simplex tableau, where the  $d_i$ 's are shadow prices, the  $v_i$ 's are activity levels, and  $z$  is the value of the objective function.

$d_1$	$d_2$	....	$d_n$	$z$
$x_{11}$	$x_{12}$		$x_{1,2n+2m+1}$	$v_1$
$x_{21}$	$x_{22}$		$x_{2,2n+2m+1}$	$v_2$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$x_{2m+1,1}$	$x_{2m+1,2}$		$x_{2m+1,2n+2m+1}$	$v_{2m+1}$

If the  $k^{\text{th}}$  column is introduced into the basis, the  $j^{\text{th}}$  column will be eliminated from the basis, where  $j$  is given by

$$v_j/x_{jk} = \min_{\substack{i=1,\dots,2m+1 \\ \text{and } x_{ik} > 0}} (v_i/x_{ik}) = \theta. \quad \text{The change in the objective}$$

function is  $-\theta d_k$ . Since  $d_k$  is negative and  $\theta \geq 0$ , the objective function will be increased only if  $\theta > 0$ . Initially, the vector of activity levels is  $B'$ . Hence the value of the objective function will be improved initially only if  $x_{ik} \leq 0$  for  $i=1,\dots,2m$  and  $x_{2m+1,k} > 0$ . (If the problem has a bounded optimum solution then  $x_{2m+1,k} > 0$  when  $x_{ik} \leq 0$  for  $i=1,\dots,2m$ .) Assuming the signs of the  $x_{ik}$ 's to be random, the probability that  $\theta > 0$  is  $2^{-2m}$  when the vector of activity levels is  $B'$ . If  $x_{ik} > 0$  initially, where  $i \leq 2m$ , then the new vector of activity levels is still  $B'$ . However, when  $x_{ik} \leq 0$  for  $i=1,\dots,2m$ , then the new vector of activity

levels will contain at least two positive elements. Thus by the same argument, the probability that the objective function will be increased at the next iteration is at least  $2^{-2m+1}$ . The same argument is applied until the optimum solution is obtained. The expected number,  $I$ , of iterations required to reach the optimum solution will thus satisfy  $2^{2m} \leq I \leq 2^{2m} + 2^{2m-1} + \dots + 2^0 < 2^{2m+1}$ , in the special case of the given vector  $B'$ . This is much larger than the  $2m+1$  or  $4m+2$  iterations usually stated as the number of iterations required to obtain the optimum solution.

The effect of a transformation of the dual problem which eliminates the zero elements of the vector  $B'$  will be considered. It is expected that the slow convergence described above will be avoided. The transformation matrix,  $C$ , is chosen in the following manner. After slack variables have been introduced, the dual problem can be represented by  $A''U'=B'$ , where  $A''=(A',E)$ ,  $E$  being the  $(2m+1) \times (2m+1)$  identity matrix. Let  $C$  be the  $(2m+1) \times (2m+1)$  matrix

$$\begin{array}{cccccc} 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & \dots & 0 & 1 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \end{array}$$

Since  $C$  is non-singular, the solution of the problem of maximizing

$$\sum_{k=1}^n (-f(\bar{z}_k)) u_k + \sum_{k=1}^n f(\bar{z}_k) u_{n+k} \quad \text{subject to the constraints } CA''U'=CB'$$

and  $U' \geq 0$  is the same as the solution of the problem of maximizing

$$\sum_{k=1}^n (-f(\bar{z}_k)) u_k + \sum_{k=1}^n f(\bar{z}_k) u_{n+k} \quad \text{subject to the constraints } A''U'=B' \text{ and}$$

$U' \geq 0$ . Each component of the vector  $CB'$  is 1.

### III. CONVERGENCE OF THE SIMPLEX METHOD

As an alternative approach to the problem of decreasing the computation time for the dual problem, several methods suggested by R.E. Quandt and H.W. Kuhn [2] for increasing the rate of convergence of the Simplex Method are being considered. Three new criteria for choosing a pivotal column (the next vector to be introduced into the basis) in the Simplex Method are described below, using the notation of Part II. Method (2) is the Dantzig criterion, presently used in the UNC linear programming routine.

a) The Greatest Unit Ascent Method. The  $j^{\text{th}}$  column is chosen as the pivotal column when  $d = \min_{s=1, \dots, 2n+2m+1} d_s$  and  $d_s < 0$ . This is the criterion

most frequently employed in the Simplex Method.

b) The Greatest Absolute Ascent Method. If  $s$  is a positive integer not greater than  $2n+2m+1$ , let  $r_s$  be the integer given by  $v_{r_s, s} / x_{r_s, s} = \min$

$\min_{i=1, \dots, 2m+1} (v_i / x_{is})$ . ( $x_{r_s, s}$  would be the pivotal element if the  $s^{\text{th}}$  column were the pivotal column.) Then the  $j^{\text{th}}$  column is chosen as pivotal column when

$$d_{j, r_j} v_{r_j, j} / s_{r_j, j} = \min_{s=1, \dots, 2n+2m+1} (d_s v_{r_s, s} / x_{r_s, s}) \text{ and } d_s < 0$$

c) The Minimum Next Choice Method. Let  $p_{tjr}$  equal 1 or 0 according as the choice of the  $j^{\text{th}}$  column as pivotal column at the  $t^{\text{th}}$  iteration causes  $d_r$  to be negative or non-negative at the  $(t+1)^{\text{st}}$  iteration. Then the  $j^{\text{th}}$  column is chosen as pivotal column at the  $t^{\text{th}}$  iteration if

$$\sum_{r=1}^{2n+2m+1} p_{tjr} = \min_{s=1, \dots, 2m+2n+1} \sum_{r=1}^{2n+2m+1} p_{tsr}$$

d) The Modified Gradient Method. The  $j^{\text{th}}$  column is chosen as pivotal column when

$$d_j^2 / \left( \sum_{i=1}^{2m+1} x_{ij}^2 + d_j^2 \right) = \max_{\substack{s=1, \dots, 2n+2m+1 \\ d_s < 0}} d_s^2 / \left( \sum_{i=1}^{2m+1} x_{is}^2 + d_s^2 \right).$$

The experiments performed by Quandt and Kuhn indicate that the criteria (b) and (d) may give better convergence than (a), while (c) gives slower convergence. The criteria (b) and (d) are being incorporated into the linear programming routine at UNC. It is possible, also, that the special structure of the problems being studied may make some other method for choosing the pivotal elements more effective.

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NORTHEAST LOUISIANA STATE COLLEGE  
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EXISTENCE OF MULTIVARIABLE LEAST SQUARES  
APPROXIMATING POLYNOMIALS

By

Daniel E. Dupree

SUMMARY

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Sufficiency conditions for the existence of multivariable least squares approximating functions are developed.

I. INTRODUCTION

The problem of existence of least squares approximating polynomials in a single variable has been resolved in great detail. A considerable amount of work has been done on this same problem for generalized approximating functions of several variables, but this work is of a highly abstract nature. Here we consider the problem for least squares approximating functions of several variables.

II. EXISTENCE

We can state the existence problem as follows:

If  $[\beta_0, X(\beta_0)], [\beta_1, X(\beta_1)], \dots, [\beta_n, X(\beta_n)]$  are  $n + 1$  pairs of values of the function  $X = X(\beta)$ , where  $\beta = (x_1, x_2, \dots, x_t)$ ,  $\beta_i = (x_{1i}, x_{2i}, \dots, x_{ti})$ , and if  $\varphi_0(\beta), \varphi_1(\beta), \dots, \varphi_N(\beta)$  are  $N + 1$  functions of  $\beta$ , under what conditions do there exist constants  $A_0, A_1, \dots, A_N$  such that

$$F(A_0, A_1, \dots, A_N) = \sum_{i=0}^n [X(\beta_i) - \sum_{j=0}^N A_j \varphi_j(\beta_i)]^2$$

is a minimum?

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**EXISTENCE OF MULTIVARIABLE LEAST SQUARES  
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**By**

**Daniel E. Dupree**

**Lemma 1:**  $F(A_0, A_1, \dots, A_N) = \sum_{i=0}^n [X(\beta_i) - \sum_{j=0}^N A_j \varphi_j(\beta_i)]^2$  is a

continuous function of its arguments.

**Proof:**  $\left| \sum_{i=0}^n [X(\beta_i) - \sum_{j=0}^N A_j \varphi_j(\beta_i)]^2 - \sum_{i=0}^n [X(\beta_i) - \sum_{j=0}^N A'_j \varphi_j(\beta_i)]^2 \right|$

$$= \left| [X(\beta_0) - \sum_{j=0}^N A_j \varphi_j(\beta_0)]^2 + [X(\beta_1) - \sum_{j=0}^N A_j \varphi_j(\beta_1)]^2 + \dots \right.$$

$$\left. \dots + [X(\beta_n) - \sum_{j=0}^N A_j \varphi_j(\beta_n)]^2 - [X(\beta_0) - \sum_{j=0}^N A'_j \varphi_j(\beta_0)]^2 \right.$$

$$\left. - [X(\beta_1) - \sum_{j=0}^N A'_j \varphi_j(\beta_1)]^2 - \dots - [X(\beta_n) - \sum_{j=0}^N A'_j \varphi_j(\beta_n)]^2 \right|$$

$$= \left| -2X(\beta_0) \sum_{j=0}^N A_j \varphi_j(\beta_0) + \left[ \sum_{j=0}^N A_j \varphi_j(\beta_0) \right]^2 - 2X(\beta_1) \sum_{j=0}^N A_j \varphi_j(\beta_1) \right.$$

$$\left. + \left[ \sum_{j=0}^N A_j \varphi_j(\beta_1) \right]^2 - \dots - 2X(\beta_n) \sum_{j=0}^N A_j \varphi_j(\beta_n) + \left[ \sum_{j=0}^N A_j \varphi_j(\beta_n) \right]^2 \right.$$

$$\left. + 2X(\beta_0) \sum_{j=0}^N A'_j \varphi_j(\beta_0) - \left[ \sum_{j=0}^N A'_j \varphi_j(\beta_0) \right]^2 + 2X(\beta_1) \sum_{j=0}^N A'_j \varphi_j(\beta_1) \right.$$

$$\begin{aligned}
& - \left[ \sum_{j=0}^N A'_j \varphi_j(\beta_1) \right]^2 + \dots + 2\chi(\beta_n) \sum_{j=0}^N A'_j \varphi_j(\beta_n) - \left[ \sum_{j=0}^N A'_j \varphi_j(\beta_n) \right]^2 \Big| \\
& = \Big| -2\chi(\beta_0) \left[ \sum_{j=0}^N A_j \varphi_j(\beta_0) - \sum_{j=0}^N A'_j \varphi_j(\beta_0) \right] - 2\chi(\beta_1) \left[ \sum_{j=0}^N A_j \varphi_j(\beta_1) \right. \\
& \quad \left. - \sum_{j=0}^N A'_j \varphi_j(\beta_1) \right] - \dots - 2\chi(\beta_n) \left[ \sum_{j=0}^N A_j \varphi_j(\beta_n) - \sum_{j=0}^N A'_j \varphi_j(\beta_n) \right] \\
& \quad + \left\{ \left[ \sum_{j=0}^N A_j \varphi_j(\beta_0) \right]^2 - \left[ \sum_{j=0}^N A'_j \varphi_j(\beta_0) \right]^2 \right\} + \left\{ \left[ \sum_{j=0}^N A_j \varphi_j(\beta_1) \right]^2 - \left[ \sum_{j=0}^N A'_j \varphi_j(\beta_1) \right]^2 \right\} \\
& \quad + \dots + \left\{ \left[ \sum_{j=0}^N A_j \varphi_j(\beta_n) \right]^2 - \left[ \sum_{j=0}^N A'_j \varphi_j(\beta_n) \right]^2 \right\} \Big| \\
& = \Big| \left[ \sum_{j=0}^N A_j \varphi_j(\beta_0) - \sum_{j=0}^N A'_j \varphi_j(\beta_0) \right] \cdot \left\{ -2\chi(\beta_0) + \left[ \sum_{j=0}^N A_j \varphi_j(\beta_0) + \sum_{j=0}^N A'_j \varphi_j(\beta_0) \right] \right\} \\
& \quad + \left[ \sum_{j=0}^N A_j \varphi_j(\beta_1) - \sum_{j=0}^N A'_j \varphi_j(\beta_1) \right] \cdot \left\{ -2\chi(\beta_1) + \left[ \sum_{j=0}^N A_j \varphi_j(\beta_1) + \sum_{j=0}^N A'_j \varphi_j(\beta_1) \right] \right\} \\
& \quad + \dots + \left[ \sum_{j=0}^N A_j \varphi_j(\beta_n) - \sum_{j=0}^N A'_j \varphi_j(\beta_n) \right] \cdot \left\{ -2\chi(\beta_n) \right. \\
& \quad \quad \left. + \left[ \sum_{j=0}^N A_j \varphi_j(\beta_n) + \sum_{j=0}^N A'_j \varphi_j(\beta_n) \right] \right\} \Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \left| \sum_{j=0}^N (A_j - A'_j) \varphi_j(\beta_0) \cdot \left\{ -2X(\beta_0) + \sum_{j=0}^N (A_j + A'_j) \varphi_j(\beta_0) \right\} \right| \\
&+ \left| \sum_{j=0}^N (A_j - A'_j) \varphi_j(\beta_1) \cdot \left\{ -2X(\beta_1) + \sum_{j=0}^N (A_j + A'_j) \varphi_j(\beta_1) \right\} \right| \\
&+ \dots + \left| \sum_{j=0}^N (A_j - A'_j) \varphi_j(\beta_n) \cdot \left\{ -2X(\beta_n) + \sum_{j=0}^N (A_j + A'_j) \varphi_j(\beta_n) \right\} \right| \\
&\leq \left| \sum_{j=0}^N (A_j - A'_j) \varphi_j(\beta_0) \right| \cdot \left\{ |2X(\beta_0)| + \left| \sum_{j=0}^N (A_j + A'_j) \varphi_j(\beta_0) \right| \right\} \\
&+ \left| \sum_{j=0}^N (A_j - A'_j) \varphi_j(\beta_1) \right| \cdot \left\{ |2X(\beta_1)| + \left| \sum_{j=0}^N (A_j + A'_j) \varphi_j(\beta_1) \right| \right\} \\
&+ \dots + \left| \sum_{j=0}^N (A_j - A'_j) \varphi_j(\beta_n) \right| \cdot \left\{ |2X(\beta_n)| + \left| \sum_{j=0}^N (A_j + A'_j) \varphi_j(\beta_n) \right| \right\} \\
&\leq \sum_{j=0}^N |A_j - A'_j| |\varphi_j(\beta_0)| \cdot \left\{ |2X(\beta_0)| + \sum_{j=0}^N |A_j + A'_j| |\varphi_j(\beta_0)| \right\} \\
&+ \sum_{j=0}^N |A_j - A'_j| |\varphi_j(\beta_1)| \cdot \left\{ |2X(\beta_1)| + \sum_{j=0}^N |A_j + A'_j| |\varphi_j(\beta_1)| \right\}
\end{aligned}$$

$$\begin{aligned}
& + \dots + \sum_{j=0}^N |A_j - A'_j| |\varphi_j(\beta_n)| \cdot \left\{ |2X(\beta_n)| + \sum_{j=0}^N |A_j + A'_j| |\varphi_j(\beta_n)| \right\} \\
& \leq \max_{0 \leq j \leq N} |A_j - A'_j| \cdot \left\{ \sum_{j=0}^N |\varphi_j(\beta_0)| [|2X(\beta_0)| + \max_{0 \leq j \leq N} |A_j + A'_j| \sum_{j=0}^N |\varphi_j(\beta_0)|] \right. \\
& \quad \left. + \sum_{j=0}^N |\varphi_j(\beta_1)| [|2X(\beta_1)| + \max_{0 \leq j \leq N} |A_j + A'_j| \sum_{j=0}^N |\varphi_j(\beta_1)|] \right. \\
& \quad \left. + \dots + \sum_{j=0}^N |\varphi_j(\beta_n)| [|2X(\beta_n)| + \max_{0 \leq j \leq N} |A_j + A'_j| \sum_{j=0}^N |\varphi_j(\beta_n)|] \right\}.
\end{aligned}$$

Hence,  $F(A_0, A_1, \dots, A_N)$  is a continuous function of its arguments.

Lemma 2: The function

$$Q(A_0, A_1, \dots, A_N) = \sum_{i=0}^n [-2X(\beta_i) \sum_{j=0}^N A_j \varphi_j(\beta_i)]$$

is a continuous function of its arguments.

Proof: 
$$\left| \sum_{i=0}^n [-2X(\beta_i) \sum_{j=0}^N A_j \varphi_j(\beta_i)] - \sum_{i=0}^n [-2X(\beta_i) \sum_{j=0}^N A'_j \varphi_j(\beta_i)] \right|$$

$$\begin{aligned}
&= \left| \sum_{i=0}^n [-2X(\beta_i) \sum_{j=0}^N A_j \varphi_j(\beta_i) + 2X(\beta_i) \sum_{j=0}^N A'_j \varphi_j(\beta_i)] \right| \\
&= \left| \sum_{i=0}^n [-2X(\beta_i) \sum_{j=0}^N (A_j - A'_j) \varphi_j(\beta_i)] \right| \leq \sum_{i=0}^n [|2X(\beta_i)| \sum_{j=0}^N |A_j - A'_j| |\varphi_j(\beta_i)|] \\
&\leq \max_{0 \leq j \leq N} |A_j - A'_j| \sum_{i=0}^n \left\{ |2X(\beta_i)| \sum_{j=0}^N |\varphi_j(\beta_i)| \right\}.
\end{aligned}$$

**Lemma 3:** If  $\sum_{i=0}^n X(\beta_i) \varphi_j(\beta_i) \neq 0$ , for some  $j$ ,  $0 \leq j \leq N$ , then

$Q(A_0, A_1, \dots, A_N)$  has non-zero extreme values on the sphere

$$A_0^2 + A_1^2 + \dots + A_N^2 = 1.$$

**Proof:** Let  $R(A_0, A_1, \dots, A_N) = A_0^2 + A_1^2 + \dots + A_N^2 - 1$  and let  $\gamma$  be an undetermined Lagrangian multiplier. Then the extreme values of  $Q(A_0, A_1, \dots, A_N)$  will occur at the zeros of the following system of equations:

$$\frac{\partial Q}{\partial A_0} + \gamma \frac{\partial R}{\partial A_0} = 0$$

$$\frac{\partial Q}{\partial A_1} + \gamma \frac{\partial R}{\partial A_1} = 0$$

⋮

$$\frac{\partial Q}{\partial A_N} + \gamma \frac{\partial R}{\partial A_N} = 0.$$

$$\text{But } Q(A_0, A_1, \dots, A_N) = \sum_{i=0}^n \left[ -2X(\beta_i) \sum_{j=0}^N A_j \varphi_j(\beta_i) \right]$$

$$= \sum_{i=0}^n \left[ -2X(\beta_i) \left\{ A_0 \varphi_0(\beta_i) + A_1 \varphi_1(\beta_i) + \dots + A_N \varphi_N(\beta_i) \right\} \right]$$

$$= \sum_{i=0}^n \left[ -2X(\beta_i) A_0 \varphi_0(\beta_i) - 2X(\beta_i) A_1 \varphi_1(\beta_i) - \dots - 2X(\beta_i) A_N \varphi_N(\beta_i) \right]$$

$$= -2A_0 \sum_{i=0}^n X(\beta_i) \varphi_0(\beta_i) - 2A_1 \sum_{i=0}^n X(\beta_i) \varphi_1(\beta_i) - \dots - 2A_N \sum_{i=0}^n X(\beta_i) \varphi_N(\beta_i).$$

Thus, the system above becomes

$$-2 \sum_{i=0}^n X(\beta_i) \varphi_0(\beta_i) + 2\gamma A_0 = 0$$

$$-2 \sum_{i=0}^n X(\beta_i) \varphi_1(\beta_i) + 2\gamma A_1 = 0$$

⋮

$$-2 \sum_{i=0}^n X(\beta_i) \varphi_N(\beta_i) + 2\gamma A_N = 0.$$

Now  $\gamma \neq 0$ , since  $\sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \neq 0$  for some  $j$ ,  $0 \leq j \leq N$ .

Therefore,  $A_j = 1/\gamma \left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]$ ,  $j = 0, 1, \dots, N$ , and

$A_0^2 + A_1^2 + \dots + A_N^2 = 1$  becomes

$$1/\gamma^2 \left[ \sum_{i=0}^n x(\beta_i) \varphi_0(\beta_i) \right]^2 + \left[ \sum_{i=0}^n x(\beta_i) \varphi_1(\beta_i) \right]^2 + \dots + \left[ \sum_{i=0}^n x(\beta_i) \varphi_N(\beta_i) \right]^2 = 1.$$

Hence,  $1/\gamma^2 \cdot \sum_{j=0}^N \left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]^2 = 1$ , or

$$\gamma^2 = \sum_{j=0}^N \left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]^2. \quad \text{Thus,}$$

$$\gamma = \pm \sqrt{\sum_{j=0}^N \left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]^2}, \quad \text{and}$$

$$A_j = 1/\gamma \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) = \pm \frac{\sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i)}{\sqrt{\sum_{j=0}^N \left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]^2}}.$$

Therefore, the extreme values of  $Q(A_0, A_1, \dots, A_N)$  are

$$Q(A_0, A_1, \dots, A_N) = \pm 2 \sum_{j=0}^N \frac{\sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i)}{\sqrt{\sum_{j=0}^N \left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]^2}} \cdot \left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]$$

$$= \pm 2 \sum_{j=0}^N \frac{\left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]^2}{\sqrt{\sum_{j=0}^N \left[ \sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \right]^2}}.$$

These extreme values are not zero if  $\sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i)$  fails to vanish

for some  $j$ ,  $0 \leq j \leq N$ .

**Theorem:** If  $\varphi_0(\beta)$ ,  $\varphi_1(\beta)$ , ...,  $\varphi_N(\beta)$  are  $N+1$  functions satisfying

$\sum_{i=0}^n x(\beta_i) \varphi_j(\beta_i) \neq 0$ , for some  $j$ ,  $0 \leq j \leq N$ , then there exists constants

$A_0, A_1, \dots, A_N$  such that

$$F(A_0, A_1, \dots, A_N) = \sum_{i=0}^n \left[ x(\beta_i) - \sum_{j=0}^N A_j \varphi_j(\beta_i) \right]^2$$

is a minimum.

**Proof:** Let  $\pi \neq 0$  be the minimum value of  $Q(A_0, A_1, \dots, A_N)$

on the unit sphere  $\sum_{j=0}^N A_j^2 = 1$ .

**Case 1:**  $\pi < 0$ . If  $\pi < 0$ , then  $\sqrt{\sum_{j=0}^N A_j^2} > 1/\pi (\alpha + 1)$ , where  $\alpha = \text{g. l. b.}$

of  $F(A_0, A_1, \dots, A_N)$ .  $\alpha$  exists since  $F \geq 0$ . Therefore,

$$\begin{aligned}
 F(A_0, A_1, \dots, A_N) &= \sum_{i=0}^n [X(\beta_i) - \sum_{j=0}^N A_j \varphi_j(\beta_i)]^2 \\
 &= \sum_{i=0}^n X^2(\beta_i) + \sum_{i=0}^n [-2X(\beta_i) \sum_{j=0}^N A_j \varphi_j(\beta_i)] \\
 &\quad + \sum_{i=0}^n [\sum_{j=0}^N A_j \varphi_j(\beta_i)]^2 \\
 &\geq \sum_{i=0}^n [-2X(\beta_i) \sum_{j=0}^N A_j \varphi_j(\beta_i)] \geq \pi \sqrt{\sum_{j=0}^N A_j^2} \\
 &> \pi (1/\pi) (\alpha + 1) = \alpha + 1 > \alpha,
 \end{aligned}$$

a contradiction. Hence, this case cannot occur.

**Case 2:**  $\pi > 0$ . We assume that

$$\sqrt{\sum_{j=0}^N A_j^2} > 1/\pi (\alpha + 1) \text{ and obtain the same}$$

contradiction as in Case 1. Thus,

$$\sqrt{\sum_{j=0}^N A_j^2} < 1/\pi (\alpha + 1) = R$$

is a closed bounded set of points  $(A_0, A_1, \dots, A_N)$  in  $N + 1$  dimensional space.  $F(A_0, A_1, \dots, A_N)$  is a continuous function of  $(A_0, A_1, \dots, A_N)$  and has a minimum value in or on the set.

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**COMPUTATION CENTER  
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**ANALYTIC DIFFERENTIATION BY COMPUTER APPLIED  
TO THE "FLAT EARTH" PROBLEM**

**By**

**James W. Hanson**

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SUMMARY

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The application of analytic differentiation by computer as applied to a simplified flat-earth calculus of variations problem is described.

SECTION I. INTRODUCTION

The application of the program for doing analytic differentiation by computer to developing Taylor's series expansions in a simplified flat-earth calculus of variations problem as set forth in Reference [1] is in progress. The length of the equations being processed and the complexity of the differentiation procedures have required the extension of many of the size limitations in the original differentiation program and the addition of several new capabilities to the basic program.

SECTION II. PROGRAM ADDITIONS AND EXTENSIONS

The following additions and extensions have now been incorporated into the original differentiation program:

1. The limit of the length of the input string has been increased from three hundred and twenty alpha-numeric characters to seven hundred and twenty characters.
2. The limit of the length of the string which may be generated by the program as a result of successive differentiations has been increased to forty-two hundred alpha-numeric characters.
3. Often two or more rows of the M-matrix (Reference 2) will be identical. Just prior to the differentiation of the expression a pass over the M-matrix has been introduced which eliminates all such

duplicate lines and replaces all references to these duplicate lines by references to the first occurrence of the line. This addition effects a marked decrease in the length of the M-matrix as well as a decrease in the time of differentiation. This simplification of the matrix structure is also a first step toward the development of algorithms which can do algebraic simplification of the mathematical expressions being handled. In this simplified form, the existence of common terms which can be factored or canceled can be more easily recognized.

4. A symbolic differential operator has been incorporated in the program so that chain differentiation can now be done. This addition was needed before advancing to the second and higher order derivatives of the equations of motion. A further extension needed here is the ability to read in the equations for the dependent variables, generated the derivatives as specified by the differential operators, and then by substitution, eliminate the operator from the basic derivative to obtain the required partial derivative in its final form.

5. To date, the bottleneck in the generation of the Taylor's series has been the writing of the program to evaluate the derivatives and solve for the unknown partial derivatives. With such long and involved equations, the work has been very tedious and susceptible to error. In an effort to speed up this portion of the work, a second version of the differentiator was written which outputs the derivative expressions in matrix format. The program to evaluate the derivatives can then be written by simply evaluating each row of the matrix in turn. This leads to an extremely simple program and eliminates a great deal of the human error.

Of course, the best solution to this problem is to entirely eliminate the programmer from this phase of the work by incorporating the differentiation procedure into an algebraic compiler which could automatically do the needed evaluations. This work is underway but is a long term project and also depends on the satisfactory completion of the final extension discussed in 4., above.

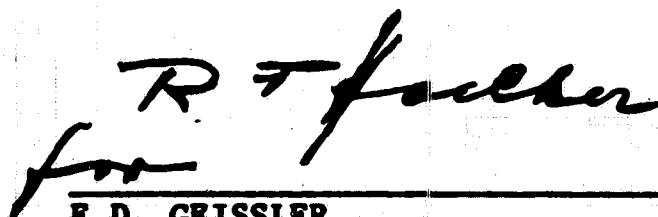
### SECTION III. PROGRESS

The program for determining the first order coefficients of the Taylor's series expansion has been completed and the coefficients computed at various points of a given trajectory. No analysis was done on the first order approximations except a simple study of their graphs.

The expressions for the second and third order derivatives have now been generated and the program for determining the numerical values for the second order coefficients is being written. Upon completion of this program, the Taylor's series of second order will be computed along a number of trajectories and a study made of the error and region of convergence.

## SECTION IV. REFERENCES

1. Hubbard, S. M., Hanson, J. W., "Analytic Differentiation by Computer Applied to the "Flat-Earth" Problem", Progress Report No. 1 on Studies in the Fields of Space Flight and Guidance Theory, MTP-AERO-61-91, NASA MSFC, 18 December 1961.
2. Hanson, J. W., Caviness, J. S., Joseph, C., "Analytic Differentiation by Computer", Computation Center, University of North Carolina, Chapel Hill, N. C., December 1961.

**APPROVAL****PROGRESS REPORT NO. 2****on Studies in the Fields of****SPACE FLIGHT AND GUIDANCE THEORY****Sponsored by Aeroballistics Division of****Marshall Space Flight Center**A handwritten signature in dark ink, appearing to read "R. T. Geissler" with a large flourish at the end. Below the signature is a horizontal line.**E.D. GEISSLER****Dir, Aeroballistics Division**